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On the Reduction of the Parametrization of all Stabilizing Controllers

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1 Introduction

It is known that the Youla-Kučera parametrization parametrizes all stabilizing controllers of a stabilizable plant whenever it admits a doubly coprime factorization [1]-[3]. If a plant does not admit a doubly coprime factorization, the Youla-Kučera parametrization is not applied to such a plant. In this case, to parametrize all stabilizaing controllers of such a plant, the general parametrization is employed. The general parametrization does not require that a plant admits a doubly coprime factorization [1], [2]. Also the general parametrization can parametrize all stabilizing controllers of a plant that admits a doubly coprime factorization. However, when the general parametrization is applied to such a plant, extra parameter variables always exist. Furthermore, when the general parametrization is applied to a plant that does not admits a doubly coprime factorization, also there exist possibility that some extra parameter variables are included in a parametrization of all stabilizing controllers of such a plant. Consequently, following two verifications are purposed in this study. (1) If a plant admits a doubly coprime factorization, by computing extra parameter variables, it is confirmed that a reduced parametrization is obtained when the general parametrization is applied to this plant. (2) If a plant does not admit a doubly coprime factorization, by computing some extra parameter variables, it is confirmed that a reduced parametrization is obtained.

2 Preliminaries

Denote by $\mathcal{A}$ a commutative ring that is the set of stable causal transfer functions. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d | n, d \in \mathcal{A}$, d: nonzero divisor$\}$. This $\mathcal{F}$ is considered to be the set of all possible transfer functions. Matrices over $\mathcal{F}$ are transfer matrices. A matrix over $\mathcal{A}$ is said to be *nonsingular* if the determinant is a nonzero divisor of $\mathcal{A}$. We consider the feedback system $\sum$ shown in Fig. 1. [1]-[3]. For details of the stabilization problem, considered this thesis, the reader is reffered to [1]-[3]. Throughout this thesis, suppose that 1) a plant has m inputs and n outputs, and its transfer matrix is denoted by P 2) P is a $n \times m$ matrix over $\mathcal{F}$ and stabilizable, 3) a transfer matrix of a stabilizing controller of P is denoted by C. Then $H(P, C)$ denotes the transfer matrix from $[u_1^t \ u_2^t]$ to $[e_1^t \ e_2^t]$ of the feedback system $\sum$, that is

$$H(P,C) := \begin{bmatrix} (I_n + PC)^{-1} & -P(I_m + CP)^{-1} \\ C(I_n + PC)^{-1} & (I_m + CP)^{-1} \end{bmatrix}$$

where $\det(I_n + PC)$ is a nonzero divisor of $\mathcal{A}$, namely $H(P, C)$ is a $(m + n) \times (m + n)$ matrix over $\mathcal{A}$ [1]. Matrices A and B over $\mathcal{A}$ are *right-* (*left-*)*coprime* if there exist matrices X and Y over $\mathcal{A}$ such that $XA + YB = I$ $(AX + BY = I)$ holds. An ordered pair (N, D) of matrices N and D over $\mathcal{F}$ is said to be a *right-coprime factorization* of P if 1) D is nonsingular, 2) $P = ND^{-1}$ over $\mathcal{F}$, 3) N and D are right-coprime. As a parallel notion, the *left - coprime factorization* of P is defined anologously. When P admits both a right-coprime factorization and left-coprime factorization, P is said to admit a *doubly coprime factorization* [1].

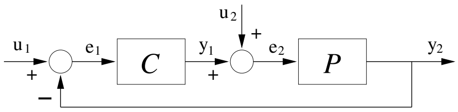

Fig. 1. Feedback system $\sum$.

3 Youla-Kučera Parametrization and General Parametrization

First, the Youla-Kučera parametrization is introduced briefly. Suppose that P admits a doubly coprime factorization. Then the set of all stabilizing controllers of P is parameterized with mn parameter variables [3].

Next, the general parametrization is introduced briefly. Suppose that C is already given one of stabilizing controllers of P. Then the set of all stabilizing controllers of P is parameterized such that $\mathcal{H}(\mathrm{P}) = \{\Omega(Q) | \Omega(Q)$ is a matrix over $\mathcal{A}$ and $\Omega(Q)$ is nonsingular$\}$ where $\Omega(Q) = \hat{\Omega}(Q) + H(P, C)$, and

$$\hat{\Omega}(Q) = \left(H(P,C) - \begin{bmatrix} I_n & O \\ O & O \end{bmatrix} \right) Q \times \left(H(P,C) - \begin{bmatrix} O & O \\ O & I_m \end{bmatrix} \right)$$

with $(m+n) \times (m+n)$ parameter matrix Q over $\mathcal{A}$ [1], [2]. Since Q has $(m + n)^2$ entries such that $Q = (q_{i,j})$ where $1 \le i, j \le m + n$, $(m + n)^2$ parameter variables exist in $\Omega(Q)$. It is possible to give an element in $\mathcal{A}$ for a parameter variable freely.

4 General Parametrization for a Polynomial Ring

We propose the reduction algorithm in the case of polynomial ring. Even so, our algorithm can be applied to the set of causal stable taransfer function is a Euclidean domain. Also, the reduction algorithm is focused on $\hat{\Omega}(Q)$ since parameter variables only exist in $\hat{\Omega}(Q)$.

In a polynomial ring, $\mathbf{R}[x]$ is $\mathcal{A}$ and $\mathbf{R}(x)$ is $\mathcal{F}$. Then $\hat{\Omega}(Q)$ is expanded and decomposed such that

$$\hat{\Omega}(Q) = A_{1,1}q_{1,1} + A_{1,2}q_{1,2} + \cdots + A_{m+n,m+n}q_{m+n,m+n}$$

where coefficient matrices $A_{1,1}$, $A_{1,2}$, $\cdots$, $A_{m+n,m+n}$ are $(m + n) \times (m + n)$ matrices over $\mathbf{R}[x]$. It is the reduction algorithm to calculate a minimal basis of $\hat{\Omega}(Q)$. By calculating a minimal basis $\hat{\Omega}(Q)$, coefficient matrices of extra parameter variables are zero matrix. Then extra parameter variables are reduced from $\hat{\Omega}(Q)$, and such reduced $\hat{\Omega}(Q)$ is denoted by $\hat{\Omega}_r(Q)$. By adding $H(P, C)$ to $\hat{\Omega}_r(Q)$, reduced $\Omega(Q)$, say $\Omega_r(Q)$, is obtained. To calculate a minimal basis, new coefficient matrix is constructed by $A_{1,1}$, $A_{1,2}$, $\cdots$, $A_{m+n,m+n}$ such that $M = \left[\vec{A}_{1,1}, \vec{A}_{1,2}, \cdots, \vec{A}_{m+n,m+n}\right]^t$

where each of row vectors corresponds to each of coefficient matrices, namely, coefficient matrices are considered as row vectors. Then a reduction starts from the first column of M. In the first column, an entry a that has the lowest degree except zero is found. There exit some quotients $q_1, \cdots, q_{m+n-1}$, and remainders $r_1, \cdots, r_{m+n-1}$ for the other entries modulo a. Note that quotients and remainders are in $\mathbf{R}[x]$. Multiplying the row vector that has a by each of these quotients with the opposite sign, namely $-q_1, \cdots, -q_{m+n-1}$, is added to each of the other row vectors. Then the entries of the other row vectors are equal to these remainders. This calculation can be reflected to $\hat{\Omega}(Q)$ since an element in $\mathbf{R}[x]$ can be given to a parameter variable freely. Namely, in $\hat{\Omega}(Q)$, the parameter variable for the coefficient matrix that has a is re-given as new prameter variable. For example, suppose that a is $(1,1)$ entry of $\vec{A}_{1,1}$, then there exit quotients $q_1, \cdots, q_{m+n-1}$ and remainders $r_1, \cdots, r_{m+n-1}$ modulo a for each of $(1,1)$ entries of $\vec{A}_{1,2}, \cdots, \vec{A}_{m+n,m+n}$. Then $q_{1,1}$ is re-given such that $\acute{q}_{1,1} = q_{1,1} - q_1 q_{1,2} - \cdots - q_{m+n-1} q_{m+n,m+n}$, and $\acute{q}_{1,1}$ is applied to $\hat{\Omega}(Q)$ as follows:

$$\begin{aligned}
\hat{\Omega}(Q) &= A_{1,1}\acute{q}_{1,1} + A_{1,2}q_{1,2} + \cdots + \\
&\quad A_{m+n,m+n}q_{m+n,m+n} \\
&= A_{1,1}(q_{1,1} - q_1 q_{1,2} - \cdots - \\
&\quad q_{m+n-1}q_{m+n,m+n}) + \\
&\quad A_{1,2}q_{1,2} + \cdots + A_{m+n,m+n}q_{m+n,m+n} \\
&= A_{1,1}q_{1,1} + (-q_1 A_{1,1} + A_{1,2})q_{1,2} + \cdots + \\
&\quad (-q_{m+n-1}A_{1,1} + A_{m+n,m+n})q_{m+n,m+n} \\
&= A_{1,1}q_{1,1} + A^*_{1,2}q_{1,2} + \cdots + \\
&\quad A^*_{m+n,m+n}q_{m+n,m+n}
\end{aligned}$$

where each of $(1,1)$ entries of $A^*_{1,2}, \cdots, A^*_{m+n,m+n}$ is each of these remainders $r_1, \cdots, r_{m+n-1}$. If all remainders are zero, a reduction in the first column of M ends since it is impossible to reduce a from $r_1, \cdots, r_{m+n-1}$. Therefore $A_{1,1}$ is one of a minimal basis, and $q_{1,1}$ is not extra parameter variables. If one or more nonzero remainders exist in r_1, $\cdots, r_{m+n-1}$, a remainder that has the lowest degree except zero is found. Then the above procedure is repeated in the first column until only one entry is nonzero and the other entries are zero. When a reduction ends in the first column, the coefficient marix that has nonzero entry is one of a minimal basis, and the parameter variable of this coefficient matrix is not extra parameter variable. Then reductions are continued from the second column to the last column of M. When a reduction in the last column ends, namely, all reductions end, $\hat{\Omega}_r(Q)$ is obtained. Note that, in a Euclidean domain, M is an upper triangular matrix by suitable interchanging row vectors when all reductions end.

Due to space limitation, we have omitted to describe the cases of classic continuous-time systems and discrete-time systems. Since these systems are also over a Euclidean domain, the reduction algorithm can be applied to these systems. For details the reader reffered to [3].

5 General Parametrization for the Case of Nonexistence of a Doubly Coprime Factorization

The polynomial ring $\mathbf{R}[d^2, d^3]$ is equal to $\mathbf{R}[d]$ except a term of first degree, namely ad^1 where $a \in \mathbf{R}$, does not exist. Moreover, P over $\mathbf{R}[d^2, d^3]$ does not admits a doubly coprime factorization. Since $\mathbf{R}[d^2, d^3]$ is not a Euclidean domain, the Euclidean division can not be employed in the reduction algorithm. Therefore, instead of the Euclidean division, we employ leading term elimination [4]. The reduction algorithem can be applied to $\mathbf{R}[d^2, d^3]$ by constructing quotients and remainders with leading term elimination.

6 Examples

As an example, we present a reduced parametrization of all stabilizing controllers of a stabilizable plant over a polynomial ring $\mathbf{R}[x]$. Both $\Omega(Q)$ and $\Omega_r(Q)$ are appeared as below.

Suppose that $P = x^2 + 1$, $C = \frac{-2}{2x^2+1}$, and $Q = (q_{i,j})$ where $1 \le i, j \le 2$. Then $\Omega(Q) = \hat{\Omega}(Q) + H(P, C)$ where

$$\begin{aligned}
H(P,C) &= \begin{pmatrix} -2x^2 - 1, & 2x^4 + 3x^2 + 1 \\ 2, & -2x^2 - 1 \end{pmatrix}, \\
\hat{\Omega}(Q) &= A_{1,1}q_{1,1} + A_{1,2}q_{1,2} + A_{2,1}q_{2,1} + A_{2,2}q_{2,2}.
\end{aligned}$$

Coefficient matrices $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, $A_{2,2}$ are as follows.

$$\begin{aligned}
A_{1,1} &= \begin{pmatrix} 4x^4 + 6x^2 + 2, & -4x^6 - 10x^4 - 8x^2 - 2 \\ -4x^2 - 2, & 4x^4 + 6x^2 + 2 \end{pmatrix}, \\
A_{1,2} &= \begin{pmatrix} -4x^2 - 4, & 4x^4 + 8x^2 + 4 \\ 4, & -4x^2 - 4 \end{pmatrix}, \\
A_{2,1} &= \\
&\begin{pmatrix} -4x^6 - 8x^4 - 5x^2 - 1, & 4x^8 + 12x^6 + 13x^4 + 6x^2 + 1 \\ 4x^4 + 4x^2 + 1, & -4x^6 - 8x^4 - 5x^2 - 1 \end{pmatrix}, \\
A_{2,2} &= \begin{pmatrix} 4x^4 + 6x^2 + 2, & -4x^6 - 10x^4 - 8x^2 - 2 \\ -4x^2 - 2, & 4x^4 + 6x^2 + 2 \end{pmatrix}.
\end{aligned}$$

While, a reduced parametrization $\Omega_r(Q)$ is such that $\Omega_r(Q) = \hat{\Omega}_r(Q) + H(P, C)$ where $\hat{\Omega}_r(Q) = A_{1,2}q_{1,2}$. $H(P, C)$ and $A_{1,2}$ are the same as $H(P, C)$ and $A_{1,2}$ of $\Omega(Q)$.

7 Conclusions

In a parametrization of all stabilizing controllers of a stabilizable plant that admits a doubly coprime factorization, extra parameter variables is reduced successfully. Hence it is possible to apply the general parametrization with the reduction algorithm to such a plant as well as the Youla-Kučera parametrization. In the case where a stabilizable plant does not admit a doubly coprime factorization, we have confirmed that some extra parameter variables can be reduced.

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