Balanced C_{12}-Bowtie Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{12} be *the* 12*-cycle*. *The* C_{12}*-bowtie* is a graph of 2 edge-disjoint C_{12}'s with a common vertex and the common vertex is called *the center* of the C_{12}-bowtie. When K_n is decomposed into edge-disjoint sum of C_{12}-bowties, we say that *K_n has a C_{12}-bowtie decomposition.* Moreover, when every vertex of K_n appears in the same number of C_{12}-bowties, we say that *K_n has a balanced C_{12}-bowtie decomposition* and this number is called *the replication number.*
In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_{12}-bowtie decomposition of K_n is $n \equiv 1 \pmod{48}$. The decomposition algorithm is also given.

2. Balanced C_{12}-bowtie decomposition of K_n

Notation. We denote a C_{12}-bowtie passing through
$v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_1$,
$v_1 - v_{13} - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_{19} - v_{20} - v_{21} - v_{22} - v_{23} - v_1$,
by
$\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}),$
$(v_1, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23})\}$.

Theorem. K_n has a balanced C_{12}-bowtie decomposition if and only if $n \equiv 1 \pmod{48}$.

Proof. **(Necessity)** Suppose that K_n has a balanced C_{12}-bowtie decomposition. Let b be the number of C_{12}-bowties and r be the replication number. Then $b = n(n-1)/48$ and $r = 23(n-1)/48$. Among r C_{12}-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{12}-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/48$ and $r_2 = 22(n-1)/48$. Therefore, $n \equiv 1 \pmod{48}$ is necessary.

(Sufficiency) Put $n = 48t + 1$. Construct tn C_{12}-bowties as follows:
$B_i^{(1)} = \{(i, i+1, i+4t+2, i+16t+2, i+28t+3, i+12t+2, i+38t+3, i+14t+2, i+32t+3, i+18t+2, i+8t+2, i+2t+1),$
$(i, i+2, i+4t+4, i+16t+3, i+28t+5, i+12t+3, i+38t+5, i+14t+3, i+32t+5, i+18t+3, i+8t+4, i+2t+2)\}$
$B_i^{(2)} = \{(i, i+3, i+4t+6, i+16t+4, i+28t+7, i+12t+4, i+38t+7, i+14t+4, i+32t+7, i+18t+4, i+8t+6, i+2t+3),$
$(i, i+4, i+4t+8, i+16t+5, i+28t+9, i+12t+5, i+38t+9, i+14t+5, i+32t+9, i+18t+5, i+8t+8, i+2t+4)\}$
$B_i^{(3)} = \{(i, i+5, i+4t+10, i+16t+6, i+28t+11, i+12t+6, i+38t+11, i+14t+6, i+32t+11, i+18t+6, i+8t+10, i+2t+5),$
$(i, i+6, i+4t+12, i+16t+7, i+28t+13, i+12t+7, i+38t+13, i+14t+7, i+32t+13, i+18t+7, i+8t+12, i+2t+6)\}$
...
$B_i^{(t)} = \{(i, i+2t-1, i+8t-2, i+18t, i+32t-1, i+14t, i+42t-1, i+16t, i+36t-1, i+20t, i+12t-2, i+4t-1),$
$(i, i+2t, i+8t, i+18t+1, i+32t+1, i+14t+1, i+42t+1, i+16t+1, i+36t+1, i+20t+1, i+12t, i+4t)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced C_{12}-bowtie decomposition of K_n.
This completes the proof.

Example 1. *Balanced C_{12}-bowtie decomposition of K_{49}.*

$B_i = \{(i, i+1, i+6, i+18, i+31, i+14, i+41, i+16, i+35, i+20, i+10, i+3),$
$(i, i+2, i+8, i+19, i+33, i+15, i+43, i+17, i+37, i+21, i+12, i+4)\} \quad (i = 1, 2, ..., 49).$

Example 2. *Balanced C_{12}-bowtie decomposition of K_{97}.*
$B_i^{(1)} = \{(i, i+1, i+10, i+34, i+59, i+26, i+79, i+30, i+67, i+38, i+18, i+5),$
$(i, i+2, i+12, i+35, i+61, i+27, i+81, i+31, i+69, i+39, i+20, i+6)\}$
$B_i^{(2)} = \{(i, i+3, i+14, i+36, i+63, i+28, i+83, i+32, i+71, i+40, i+22, i+7),$
$(i, i+4, i+16, i+37, i+65, i+29, i+85, i+33, i+73, i+41, i+24, i+8)\} \quad (i = 1, 2, ..., 97).$

Example 3. *Balanced C_{12}-bowtie decomposition of K_{145}.*
$B_i^{(1)} = \{(i, i+1, i+14, i+50, i+87, i+38, i+117, i+44, i+99, i+56, i+26, i+7),$
$(i, i+2, i+16, i+51, i+89, i+39, i+119, i+45, i+101, i+57, i+28, i+8)\}$
$B_i^{(2)} = \{(i, i+3, i+18, i+52, i+91, i+40, i+121, i+46, i+103, i+58, i+30, i+9),$
$(i, i+4, i+20, i+53, i+93, i+41, i+123, i+47, i+105, i+59, i+32, i+10)\}$
$B_i^{(3)} = \{(i, i+5, i+22, i+54, i+95, i+42, i+125, i+48, i+107, i+60, i+34, i+11),$
$(i, i+6, i+24, i+55, i+97, i+43, i+127, i+49, i+109, i+61, i+36, i+12)\} \quad (i = 1, 2, ..., 145).$

Example 4. *Balanced C_{12}-bowtie decomposition of K_{193}.*
$B_i^{(1)} = \{(i, i+1, i+18, i+66, i+115, i+50, i+155, i+58, i+131, i+74, i+34, i+9),$
$(i, i+2, i+20, i+67, i+117, i+51, i+157, i+59, i+133, i+74, i+36, i+10)\}$
$B_i^{(2)} = \{(i, i+3, i+22, i+68, i+119, i+52, i+159, i+60, i+135, i+76, i+38, i+11),$
$(i, i+4, i+24, i+69, i+121, i+53, i+161, i+61, i+137, i+77, i+40, i+12)\}$
$B_i^{(3)} = \{(i, i+5, i+26, i+70, i+123, i+54, i+163, i+62, i+139, i+78, i+42, i+13),$
$(i, i+6, i+28, i+71, i+125, i+55, i+165, i+63, i+141, i+79, i+44, i+14)\}$
$B_i^{(4)} = \{(i, i+7, i+30, i+72, i+127, i+56, i+167, i+64, i+143, i+80, i+46, i+15),$
$(i, i+8, i+32, i+73, i+129, i+57, i+169, i+65, i+145, i+81, i+48, i+16)\} \quad (i = 1, 2, ..., 193).$

Example 5. *Balanced C_{12}-bowtie decomposition of K_{241}.*
$B_i^{(1)} = \{(i, i+1, i+22, i+82, i+143, i+62, i+193, i+72, i+163, i+92, i+42, i+11),$
$(i, i+2, i+24, i+83, i+145, i+63, i+195, i+73, i+165, i+93, i+44, i+12)\}$
$B_i^{(2)} = \{(i, i+3, i+26, i+84, i+147, i+64, i+197, i+74, i+167, i+94, i+46, i+13),$
$(i, i+4, i+28, i+85, i+149, i+65, i+199, i+75, i+169, i+95, i+48, i+14)\}$
$B_i^{(3)} = \{(i, i+5, i+30, i+86, i+151, i+66, i+201, i+76, i+171, i+96, i+50, i+15),$
$(i, i+6, i+32, i+87, i+153, i+67, i+203, i+77, i+173, i+97, i+52, i+16)\}$
$B_i^{(4)} = \{(i, i+7, i+34, i+88, i+155, i+68, i+205, i+78, i+175, i+98, i+54, i+17),$
$(i, i+8, i+36, i+89, i+157, i+69, i+207, i+79, i+177, i+99, i+56, i+18)\}$
$B_i^{(5)} = \{(i, i+9, i+38, i+90, i+159, i+70, i+209, i+80, i+179, i+100, i+58, i+19),$
$(i, i+10, i+40, i+91, i+161, i+71, i+211, i+81, i+181, i+101, i+60, i+20)\} \quad (i = 1, 2, ..., 241).$

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