Balanced C_7-Trefoil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_7 be *the 7-cycle*. *The C_7-trefoil* is a graph of 3 edge-disjoint C_7's with a common vertex and the common vertex is called *the center of the C_7-trefoil*. When K_n is decomposed into edge-disjoint sum of C_7-trefoils, we say that K_n *has a C_7-trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of C_7-trefoils, we say that K_n *has a balanced C_7-trefoil decomposition* and this number is called *the replication number*.

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_7-trefoil decomposition of K_n is $n \equiv 1 \pmod{42}$. The decomposition algorithm is also given.

2. Balanced C_7-trefoil decomposition of K_n

Notation. We denote a C_7-trefoil passing through
$v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_1$, $v_1 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_1$, $v_1 - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_{19} - v_1$, by
$\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7), (v_1, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}), (v_1, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19})\}$.

Theorem. K_n has a balanced C_7-trefoil decomposition if and only if $n \equiv 1 \pmod{42}$.

Proof. **(Necessity)** Suppose that K_n has a balanced C_7-trefoil decomposition. Let b be the number of C_7-trefoils and r be the replication number. Then $b = n(n-1)/42$ and $r = 19(n-1)/42$. Among r C_7-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_7-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/42$ and $r_2 = 3(n-1)/7$. Therefore, $n \equiv 1 \pmod{42}$ is necessary.

(Sufficiency) Put $n = 42t + 1$. We consider 5 cases.

Case 1. $t = 1$ **and** $n = 43$. **(Example 1.)**
Construct 43 C_7-trefoils as follows:
$B_i = \{(i, i+1, i+5, i+20, i+30, i+39, i+16),$
$(i, i+2, i+7, i+21, i+32, i+40, i+18),$
$(i, i+3, i+9, i+22, i+34, i+41, i+17)\}$
$(i = 1, 2, ..., 43)$.

Case 2. $t \equiv 2 \pmod 4$**,** $t \geq 2$**, and** $n = 42t+1$**.**
Put $t = 4p + 2$. Then $3t = 12p + 6$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{3t}$ with
$S_1 : g_1, g_3, g_5, ..., g_{6p+1}$
$S_2 : g_2, g_4, g_6, ..., g_{6p+2}$
$S_3 : g_{6p+3}$
$S_4 : g_{6p+4}, g_{6p+5}, g_{6p+6}, ..., g_{3t}$ such as
$S_1 : 18t-2, 18t-4, 18t-6, ..., 18t-6p-2$
$S_2 : 18t+1, 18t-1, 18t-3, ..., 18t-6p+1$
$S_3 : 18t-6p-1$
$S_4 : 18t-6p-3, 18t-6p-4, 18t-6p-5, ..., 15t+1$.
Construct tn C_7-trefoils as follows:
$B_i^{(1)} = \{ (i, i+1, i+3t+2, i+18t+2, i+27t+3, i+36t+3, i+g_1),$
$(i, i+2, i+3t+4, i+18t+3, i+27t+5, i+36t+4, i+g_2),$
$(i, i+3, i+3t+6, i+18t+4, i+27t+7, i+36t+5, i+g_3)\}$
$B_i^{(2)} = \{ (i, i+4, i+3t+8, i+18t+5, i+27t+9, i+36t+6, i+g_4),$
$(i, i+5, i+3t+10, i+18t+6, i+27t+11, i+36t+7, i+g_5),$
$(i, i+6, i+3t+12, i+18t+7, i+27t+13, i+36t+8, i+g_6)\}$
...,
$B_i^{(t)} = \{ (i, i+3t-2, i+9t-4, i+21t-1, i+$

$33t-3, i+39t, i+g_{3t-2})$,
$(i, i+3t-1, i+9t-2, i+21t, i+33t-1, i+39t+1, i+g_{3t-1})$,
$(i, i+3t, i+9t, i+21t+1, i+33t+1, i+39t+2, i+g_{3t})\}\quad (i=1,2,...,n)$.

Case 3. $t \equiv 3 \pmod 4$, $t \geq 3$, and $n = 42t+1$.
Put $t = 4p+3$. Then $3t = 12p+9$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{3t}$ with
$S_1 : g_1, g_{6p+5}, g_{3t-2}, g_{3t-1}, g_{3t}$
$S_2 : g_2, g_3, g_4, ..., g_{6p+3}$
$S_3 : g_{6p+4}, g_{6p+6}, g_{6p+8}, \cdots, g_{3t-3}$
$S_4 : g_{6p+7}, g_{6p+9}, g_{6p+11}, \ldots, g_{3t-4}$ such as
$S_1 : 18t+1, 18t-6p-3, 15t+5, 15t+3, 15t+1$
$S_2 : 18t-1, 18t-2, 18t-3, ..., 18t-6p-2$
$S_3 : 18t-6p-5, 18t-6p-7, 18t-6p-9, ..., 15t+2$
$S_4 : 18t-6p-4, 18t-6p-6, 18t-6p-8, ..., 15t+7$.
Construct tn C_7-trefoils like Case 2.

Case 4. $t \equiv 0 \pmod 4$, $t \geq 4$, and $n = 42t+1$.
Put $t = 4p$. Then $3t = 12p$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{3t}$ with
$S_1 : g_1, g_3, g_5, ..., g_{6p-3}$
$S_2 : g_2, g_4, g_6, ..., g_{6p}$
$S_3 : g_{6p-1}$
$S_4 : g_{6p+1}, g_{6p+2}, g_{6p+3}, \ldots, g_{3t}$ such as
$S_1 : 18t-2, 18t-4, 18t-6, ..., 18t-6p+2$
$S_2 : 18t+1, 18t-1, 18t-3, ..., 18t-6p+3$
$S_3 : 18t-6p+1$
$S_4 : 18t-6p, 18t-6p-1, 18t-6p-2, ..., 15t+1$.
Construct tn C_7-trefoils like Case 2.

Case 5. $t \equiv 1 \pmod 4$, $t \geq 5$, and $n = 42t+1$.
Put $t = 4p+1$. Then $3t = 12p+3$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{3t}$ with
$S_1 : g_1, g_{6p+1}, g_{3t-2}, g_{3t-1}, g_{3t}$
$S_2 : g_2, g_3, g_4, ..., g_{6p}$
$S_3 : g_{6p+2}, g_{6p+4}, g_{6p+6}, \cdots, g_{3t-3}$
$S_4 : g_{6p+3}, g_{6p+5}, g_{6p+7}, \ldots, g_{3t-4}$ such as
$S_1 : 18t+1, 18t-6p-1, 15t+5, 15t+3, 15t+1$
$S_2 : 18t-1, 18t-2, 18t-3, ..., 18t-6p+1$
$S_3 : 18t-6p-3, 18t-6p-5, 18t-6p-7, ..., 15t+2$
$S_4 : 18t-6p, 18t-6p-2, 18t-6p-4, ..., 15t+7$.
Construct tn C_7-trefoils like Case 2.
Then they comprise a balanced C_7-trefoil decomposition of K_n.
This completes the proof.

Example 2. *Balanced C_7-trefoil decomposition of K_{85}.*
$B_i^{(1)} = \{(i, i+1, i+8, i+38, i+57, i+75, i+34)$,
$(i, i+2, i+10, i+39, i+59, i+76, i+37)$,
$(i, i+3, i+12, i+40, i+61, i+77, i+35)\}$
$B_i^{(2)} = \{(i, i+4, i+14, i+41, i+63, i+78, i+33)$,
$(i, i+5, i+16, i+42, i+65, i+79, i+32)$,
$(i, i+6, i+18, i+43, i+67, i+80, i+31)\}$
$(i = 1, 2, ..., 85)$.

Example 3. *Balanced C_7-trefoil decomposition of K_{127}.*
$B_i^{(1)} = \{(i, i+1, i+11, i+56, i+84, i+111, i+55)$,
$(i, i+2, i+13, i+57, i+86, i+112, i+53)$,
$(i, i+3, i+15, i+58, i+88, i+113, i+52)\}$
$B_i^{(2)} = \{(i, i+4, i+17, i+59, i+90, i+114, i+49)$,
$(i, i+5, i+19, i+60, i+92, i+115, i+51)$,
$(i, i+6, i+21, i+61, i+94, i+116, i+47)\}$
$B_i^{(3)} = \{(i, i+7, i+23, i+62, i+96, i+117, i+50)$,
$(i, i+8, i+25, i+63, i+98, i+118, i+48)$,
$(i, i+9, i+27, i+64, i+100, i+119, i+46)\}$
$(i = 1, 2, ..., 127)$.

Example 4. *Balanced C_7-trefoil decomposition of K_{169}.*
$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+70)$,
$(i, i+2, i+16, i+75, i+113, i+148, i+73)$,
$(i, i+3, i+18, i+76, i+115, i+149, i+68)\}$
$B_i^{(2)} = \{(i, i+4, i+20, i+77, i+117, i+150, i+71)$,
$(i, i+5, i+22, i+78, i+119, i+151, i+67)$,
$(i, i+6, i+24, i+79, i+121, i+152, i+69)\}$
$B_i^{(3)} = \{(i, i+7, i+26, i+80, i+123, i+153, i+66)$,
$(i, i+8, i+28, i+81, i+125, i+154, i+65)$,
$(i, i+9, i+30, i+82, i+127, i+155, i+64)\}$
$B_i^{(4)} = \{(i, i+10, i+32, i+83, i+129, i+156, i+63)$,
$(i, i+11, i+34, i+84, i+131, i+157, i+62)$,
$(i, i+12, i+36, i+85, i+133, i+158, i+61)\}$
$(i = 1, 2, ..., 169)$.

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