Balanced C_7-Bowtie Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_7 be *the 7-cycle*. *The C_7-bowtie* is a graph of 2 edge-disjoint C_7's with a common vertex and the common vertex is called *the center of the C_7-bowtie*. When K_n is decomposed into edge-disjoint sum of C_7-bowties, we say that K_n *has a C_7-bowtie decomposition*. Moreover, when every vertex of K_n appears in the same number of C_7-bowties, we say that K_n *has a balanced C_7-bowtie decomposition* and this number is called *the replication number*.

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_7-bowtie decomposition of K_n is $n \equiv 1 \pmod{28}$. The decomposition algorithm is also given.

2. Balanced C_7-bowtie decomposition of K_n

Notation. We denote a C_7-bowtie passing through
$v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_1$, $v_1 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_1$, by
$\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7), (v_1, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13})\}$.

Theorem. K_n has a balanced C_7-bowtie decomposition if and only if $n \equiv 1 \pmod{28}$.

Proof. **(Necessity)** Suppose that K_n has a balanced C_7-bowtie decomposition. Let b be the number of C_7-bowties and r be the replication number. Then $b = n(n-1)/28$ and $r = 13(n-1)/28$. Among r C_7-bowties having a vertex v of K_n, let r_1 and r_2 be the numbers of C_7-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/28$ and $r_2 = 3(n-1)/7$. Therefore, $n \equiv 1 \pmod{28}$ is necessary.

(Sufficiency) Put $n = 28t + 1$. We consider 4 cases.

Case 1. $t = 1$ **and** $n = 29$. **(Example 1.)**
Construct 29 C_7-bowties as follows:
$B_i = \{(i, i+1, i+4, i+14, i+21, i+27, i+13), (i, i+2, i+6, i+15, i+23, i+28, i+11)\}$
$(i = 1, 2, ..., 29)$.

Case 2. $t = 2$ **and** $n = 57$. **(Example 2.)**
Construct 114 C_7-bowties as follows:
$B_i^{(1)} = \{(i, i+1, i+6, i+26, i+39, i+51, i+23), (i, i+2, i+8, i+27, i+41, i+52, i+25)\}$
$B_i^{(2)} = \{(i, i+3, i+10, i+28, i+43, i+53, i+22), (i, i+4, i+12, i+29, i+45, i+54, i+21)\}$
$(i = 1, 2, ..., 57)$.

Case 3. t**:odd,** $t \geq 3$**, and** $n = 28t + 1$.
Put $t = 2p + 1$. Then $2t = 4p + 2$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{2t}$ with
$S_1 : g_1, g_3, g_5, ..., g_{2p-1}$
$S_2 : g_2, g_4, g_6, ..., g_{2p}$
$S_3 : g_{2p+1}$
$S_4 : g_{2p+2}, g_{2p+3}, g_{2p+4}, ..., g_{2t}$ such as
$S_1 : 12t-2, 12t-4, 12t-6, ..., 11t+1$
$S_2 : 12t+1, 12t-1, 12t-3, ..., 11t+4$
$S_3 : 11t+2$
$S_4 : 11t, 11t-1, 11t-2, ..., 10t+1$.
Construct tn C_7-bowties as follows:
$B_i^{(1)} = \{(i, i+1, i+2t+2, i+12t+2, i+18t+3, i+24t+3, i+g_1),$
$(i, i+2, i+2t+4, i+12t+3, i+18t+5, i+24t+4, i+g_2)\}$
$B_i^{(2)} = \{(i, i+3, i+2t+6, i+12t+4, i+18t+7, i+24t+5, i+g_3),$
$(i, i+4, i+2t+8, i+12t+5, i+18t+9, i+24t+6, i+g_4)\}$
$B_i^{(3)} = \{(i, i+5, i+2t+10, i+12t+6, i+18t+11, i+24t+7, i+g_5),$

$(i, i+6, i+2t+12, i+12t+7, i+18t+13, i+24t+8, i+g_6)\}$
...
$B_i^{(t)} = \{(i, i+2t-1, i+6t-2, i+14t, i+22t-1, i+26t+1, i+g_{2t-1}),$
$(i, i+2t, i+6t, i+14t+1, i+22t+1, i+26t+2, i+g_{2t})\}$
$(i = 1, 2, ..., n)$.

Case 4. t:even, $t \geq 4$, and $n = 28t+1$.
Put $t = 2p+2$. Then $2t = 4p+4$.
Consider a sequence $S : g_1, g_2, g_3, ..., g_{2t}$ with
$S_1 : g_1, g_3, g_5, ..., g_{2p-1}$
$S_2 : g_2, g_4, g_6, ..., g_{2p+2}$
$S_3 : g_{2p+1}$
$S_4 : g_{2p+3}, g_{2p+4}, g_{2p+5}, ..., g_{2t}$ such as
$S_1 : 12t-2, 12t-4, 12t-6, ..., 11t+2$
$S_2 : 12t+1, 12t-1, 12t-3, ..., 11t+3$
$S_3 : 11t+1$
$S_4 : 11t, 11t-1, 11t-2, ..., 10t+1$.
Construct tn C_7-bowties by the same way as Case 3.
Then they comprise a balanced C_7-bowtie decomposition of K_n.
This completes the proof.

Example 3. *Balanced C_7-bowtie decomposition of K_{85}.*
$B_i^{(1)} = \{(i, i+1, i+8, i+38, i+57, i+75, i+34),$
$(i, i+2, i+10, i+39, i+59, i+76, i+37)\}$
$B_i^{(2)} = \{(i, i+3, i+12, i+40, i+61, i+77, i+35),$
$(i, i+4, i+14, i+41, i+63, i+78, i+33)\}$
$B_i^{(3)} = \{(i, i+5, i+16, i+42, i+65, i+79, i+32),$
$(i, i+6, i+18, i+43, i+67, i+80, i+31)\}$
$(i = 1, 2, ..., 85)$.

Example 4. *Balanced C_7-bowtie decomposition of K_{113}.*
$B_i^{(1)} = \{(i, i+1, i+10, i+50, i+75, i+99, i+46),$
$(i, i+2, i+12, i+51, i+77, i+100, i+49)\}$
$B_i^{(2)} = \{(i, i+3, i+14, i+52, i+79, i+101, i+45),$
$(i, i+4, i+16, i+53, i+81, i+102, i+47)\}$
$B_i^{(3)} = \{(i, i+5, i+18, i+54, i+83, i+103, i+44),$
$(i, i+6, i+20, i+55, i+85, i+104, i+43)\}$
$B_i^{(4)} = \{(i, i+7, i+22, i+56, i+87, i+105, i+42),$
$(i, i+8, i+24, i+57, i+89, i+106, i+41)\}$
$(i = 1, 2, ..., 113)$.

Example 5. *Balanced C_7-bowtie decomposition of K_{141}.*
$B_i^{(1)} = \{(i, i+1, i+12, i+62, i+93, i+123, i+58),$
$(i, i+2, i+14, i+63, i+95, i+124, i+61)\}$
$B_i^{(2)} = \{(i, i+3, i+16, i+64, i+97, i+125, i+56),$
$(i, i+4, i+18, i+65, i+99, i+126, i+59)\}$
$B_i^{(3)} = \{(i, i+5, i+20, i+66, i+101, i+127, i+57),$
$(i, i+6, i+22, i+67, i+103, i+128, i+55)\}$
$B_i^{(4)} = \{(i, i+7, i+24, i+68, i+105, i+129, i+54),$
$(i, i+8, i+26, i+69, i+107, i+130, i+53)\}$
$B_i^{(5)} = \{(i, i+9, i+28, i+70, i+109, i+131, i+52),$
$(i, i+10, i+30, i+71, i+111, i+132, i+51)\}$
$(i = 1, 2, ..., 141)$.

Example 6. *Balanced C_7-bowtie decomposition of K_{169}.*
$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+70),$
$(i, i+2, i+16, i+75, i+113, i+148, i+73)\}$
$B_i^{(2)} = \{(i, i+3, i+18, i+76, i+115, i+149, i+68),$
$(i, i+4, i+20, i+77, i+117, i+150, i+71)\}$
$B_i^{(3)} = \{(i, i+5, i+22, i+78, i+119, i+151, i+67),$
$(i, i+6, i+24, i+79, i+121, i+152, i+69)\}$
$B_i^{(4)} = \{(i, i+7, i+26, i+80, i+123, i+153, i+66),$
$(i, i+8, i+28, i+81, i+125, i+154, i+65)\}$
$B_i^{(5)} = \{(i, i+9, i+30, i+82, i+127, i+155, i+64),$
$(i, i+10, i+32, i+83, i+129, i+156, i+63)\}$
$B_i^{(6)} = \{(i, i+11, i+34, i+84, i+131, i+157, i+62),$
$(i, i+12, i+36, i+85, i+133, i+158, i+61)\}$
$(i = 1, 2, ..., 169)$.

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