A Note on the Complexity of Scheduling for Precedence Constrained Messages in Distributed Systems *

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1 Introduction

This note considers a problem of minimum length scheduling for a set of messages subject to precedence constraints for switching and communication networks. The problem was first studied by Barcaccia, Bonuccelli, and Di Iannii [1].

We consider a network with n inputs and n outputs. The messages to be sent are represented by an $n \times n$ matrix $D = [d_{ij}]$, the traffic matrix, with nonnegative integer entries. Entry d_{ij} represents the number of messages to be sent from input i to output j. In order to specify precedence constraints among messages, we represent a traffic matrix D by a sequence of $n \times n$ matrices $\mathbf{D} = (D^{(1)}, D^{(2)}, \ldots, D^{(k)})$ such that $D = \sum_{i=1}^{k} D^{(i)}$. We consider precedence constraints on the rows, which means that the entries in each row of $D^{(i+1)}$ can be scheduled only if the entries in the corresponding row of $D^{(i)}$ have already been scheduled ($1 \leq i \leq k-1$).

A switching matrix is a binary matrix with at most one nonzero entry in each row and in each column. A switching matrix represents messages that can be sent simultaneously without conflicts.

A sequence of $n \times n$ switching matrices $\mathbf{S} = (S^{(1)}, S^{(2)}, \ldots, S^{(t)})$ is called a switching schedule for $\mathbf{D}$ if the following conditions are satisfied:

(1)
$$\sum_{i=1}^{t} S^{(i)} = \sum_{i=1}^{k} D^{(i)} = D;$$

(2) For any integers p, $1 \leq p \leq k$, and i, $1 \leq i \leq n$, there exists an integer q, $1 \leq q \leq t$, such that

$$\sum_{r=1}^{q} s_{ij}^{(r)} = \sum_{r=1}^{p} d_{ij}^{(r)}$$

holds for every j, $1 \leq j \leq n$.

Notice that condition (2) corresponds to the precedence constraints on the rows. Integer t is called the length of $\mathbf{S}$ and denoted by $|\mathbf{S}|$.

We consider the following problems.

Problem 1 (PCRMS) *Given* $\mathbf{D} = (D^{(1)}, D^{(2)}, \ldots, D^{(k)})$ *and positive integer* h*, decide if there exists a switching schedule* $\mathbf{S}$ *for* $\mathbf{D}$ *with* $|\mathbf{S}| \leq h$. ■

Problem 2 (MIN-PCRMS-k) *Given* $\mathbf{D} = (D^{(1)}, D^{(2)}, \ldots, D^{(k)})$*, find a switching schedule* $\mathbf{S}$ *for* $\mathbf{D}$ *with minimum length.* ■

It is shown in [1] that PCRMS is NP-complete if $k = 2$, $D^{(1)}$ is a binary matrix and $D^{(2)}$ is a ternary matrix, and $h = 3$. We improve this by showing the following .

Theorem 1 PCRMS *is* NP*-complete if* $k = 2$*,* $D^{(1)}$ *and* $D^{(2)}$ *are binary matrices, and* $h = 3$. ■

It should be noted that PCRMS can be solved in polynomial time if $k = 1$ or $h \leq 2$.

It follows from Theorem 1 that even MIN-PCRMS-2 is NP-hard. It is proved in [1] that for any positive integer k and positive number $\epsilon < 4/3$, there exists no polynomial time ϵ-approximation algorithm for MIN-PCRMS-k unless P = NP. It is also mentioned in [1] that the following naive algorithm is a polynomial time k-approximation algorithm for MIN-PCRMS-k.

Algorithm 1

Step 1: Find an optimal switching schedule for $D^{(i)}$ ($1 \leq i \leq k$).

Step 2: Schedule $D^{(i+1)}$ after the schedule for $D^{(i)}$ ($1 \leq i \leq k-1$). ■

Thus, the approximation ratio of a polynomial time approximation algorithm for MIN-PCRMS-k is between $4/3$ and k if $k \geq 2$.

We show an estimate of the approximation ratio of Algorithm 1 by means of the structure of $\mathbf{D}$. For an $n \times n$ matrix $M = [m_{ij}]$, define that

$$L(M) = \max\left\{ \sum_{k=1}^{n} m_{ik}, \sum_{k=1}^{n} m_{kj} \middle| 1 \leq i, j \leq n \right\},$$
$$l(M) = \min\left\{ \sum_{k=1}^{n} m_{ik}, \sum_{k=1}^{n} m_{kj} \middle| 1 \leq i, j \leq n \right\}.$$

For $\mathbf{D} = (D^{(1)}, D^{(2)}, \ldots, D^{(k)})$, define that

$$\alpha(\mathbf{D}) = \min\left\{ \frac{l(D^{(i)})}{L(D^{(i)})} \middle| 1 \leq i \leq k \right\},$$
$$\beta(\mathbf{D}) = \max\left\{ \frac{l(D^{(i)})}{L(D^{(i)})} \middle| 1 \leq i \leq k \right\}.$$

Theorem 2 *The approximation ratio of* Algorithm 1 *for* MIN-PCRMS-k *is at most* $2 - \beta(\mathbf{D})$ *if* $k = 2$*, and at most* $k - (k-1)\alpha(\mathbf{D})$ *if* $k \geq 3$. ■

Theorem 3 *The approximation ratio of* Algorithm 1 *for* MIN-PCRMS-k *is at least* $k - (k-1)\beta(\mathbf{D})$ *for any positive integer* k. ■

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*分散システムの先行制約を考慮した通信スケジュールの計算複雑度について
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2 Proof of Theorem 1

We first need some preliminaries. Let $B = (X, Y, E)$ be a bipartite graph with maximum vertex degree 3, where (X, Y) is a bipartition of B, and E is the set of edges of B. We denote by X^δ and Y^δ the sets of vertices in X and Y with degree δ, respectively. Let E_1 be a perfect matching of B, and E_2 be a perfect matching of $(X', Y', E - E_1)$, where X' and Y' denote the sets of nonisolated vertices in X and Y, respectively, after the removal of the edges in E_1. (E_1, E_2) is called a double perfect matching for B. It is mentioned in [1] that the following problem is NP-complete:

Problem 3 (DPM-3) *Given a bipartite graph* $B = (X, Y, E)$ *with maximum vertex degree* 3*, and* $|X^\delta| = |Y^\delta|$ $(1 \le \delta \le 3)$*, decide if there exists a double perfect matching for* B*.* ∎

Now we are ready to prove the theorem. It is obvious that our problem is in NP. We prove the theorem by showing a polynomial time reduction from DPM-3 to PCRMS.

Let $B = (X, Y, E)$ be a bipartite graph as an instance of DPM-3. Let $X = \{x_1, \ldots, x_n\}$, $X^1 = \{x_1, \ldots, x_{n_1}\}$, $X^2 = \{x_{n_1+1}, \ldots, x_{n_1+n_2}\}$, $Y = \{y_1, \ldots, y_n\}$, $Y^1 = \{y_1, \ldots, y_{n_1}\}$, and $Y^2 = \{y_{n_1+1}, \ldots, y_{n_1+n_2}\}$. We assume without loss of generality that $n_1 \neq 1$.

For any $F \subseteq X \times Y$, $M(F) = [m_{ij}]$ is an $n \times n$ binary matrix defined as:

$$m_{ij} = \begin{cases} 1 & \text{if } (x_i, y_j) \in F, \\ 0 & \text{otherwise.} \end{cases}$$

M is considered as a bijection from $2^{X \times Y}$ to the set of $n \times n$ binary matrices.

We define matrices $D^{(1)}$ and $D^{(2)}$ as follows: $D^{(1)} = M(E)$; $D^{(2)} = D'^{(2)} + D''^{(2)}$ where $D'^{(2)} = [d'^{(2)}_{ij}]$ and $D'^{(2)} = [d'^{(2)}_{ij}]$ are binary matrices defined as

$$d'^{(2)}_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \le n_1 \text{ or } (i, j) = (n_1, 1), \\ 0 & \text{otherwise;} \end{cases}$$

$$d''^{(2)}_{ij} = \begin{cases} 1 & \text{if } i = j \le n_1 + n_2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|X^\delta| = |Y^\delta| = n_\delta$ $(\delta = 1, 2)$, $L(D^{(1)} + D^{(2)}) = l(D^{(1)} + D^{(2)}) = 3$. It is easy to see that binary matrices $D^{(1)}$ and $D^{(2)}$ can be constructed in polynomial time.

We will prove that there exists a double perfect matching (E_1, E_2) for B if and only if there exists a switching schedule $\mathbf{S}$ for $\mathbf{D} = (D^{(1)}, D^{(2)})$ with $|\mathbf{S}| = 3$.

If there exists a double perfect matching (E_1, E_2) for B, then $(M(E_1), M(E_2) + D'^{(2)}, M(E - (E_1 \cup E_2)) + D''^{(2)})$ is a switching schedule for D with length 3.

Conversely, if there exists a switching schedule $\mathbf{S} = (S^{(1)}, S^{(2)}, S^{(3)})$ for $\mathbf{D}$, then $(M^{-1}(S^{(1)}), M^{-1}(QS^{(2)}))$ is a double perfect matching for B, where $Q = [q_{ij}]$ is an $n \times n$ binary matrix defined as

$$q_{ij} = \begin{cases} 1 & \text{if } i = j \ge n_1 + 1, \\ 0 & \text{otherwise.} \end{cases}$$

3 Proof of Theorem 2

Let $L_i = L(D^{(i)})$ and $l_i = l(D^{(i)})$, $1 \le i \le k$, and ρ be the approximation ration of Algorithm 1. It is easy to see that

$$\rho \le \frac{L_1 + L_2 + \cdots + L_k}{\max_i\{L_i + \sum_{j \neq i} l_i\}}.$$

We first consider the case when $k = 2$. Assume without loss of generality that $\rho(\mathbf{D}) = l(D^{(1)})/L(D^{(1)})$. We distinguish two cases.

(i) If $L_1 + l_2 \le l_1 + L_2$ then we have

$$\begin{aligned} \rho &\le \frac{L_1 + L_2}{l_1 + L_2} = 1 + \frac{L_1 - l_1}{l_1 + L_2} \le 1 + \frac{L_1 - l_1}{L_1 + l_2} \\ &\le 1 + \frac{L_1 - l_1}{L_1} = 2 - \beta(\mathbf{D}). \end{aligned}$$

(ii) If $L_1 + l_2 > l_1 + L_2$ then we have

$$\begin{aligned} \rho &\le \frac{L_1 + L_2}{L_1 + l_2} = 1 + \frac{L_2 - l_2}{L_1 + l_2} < 1 + \frac{L_1 - l_1}{L_1 + l_2} \\ &\le 1 + \frac{L_1 - l_1}{L_1} = 2 - \beta(\mathbf{D}). \end{aligned}$$

We next consider the case when $k \ge 3$. Assume without loss of generality that $\max_i\{L_i + \sum_{j \neq i} l_i\} = L_1 + l_2 + \cdots + l_k$. Then, we have $L_1 + l_i \ge l_1 + L_i$ for any $i \ge 2$, and

$$\begin{aligned} \rho &\le 1 + \frac{\sum_{i=2}^{k}(L_i - l_i)}{L_1 + l_2 + \cdots + l_k} \le 1 + \frac{(k-1)(L_1 - l_1)}{L_1 + l_2 + \cdots + l_k} \\ &\le 1 + \frac{(k-1)(L_1 - l_1)}{L_1} = k - (k-1)\alpha(\mathbf{D}). \end{aligned}$$

4 Proof of Theorem 3

Considering $\mathbf{D} = (D^{(1)}, D^{(2)}, \ldots, D^{(k)})$ defined as:

$$d^{(1)}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i = 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$d^{(r)}_{ij} = \begin{cases} 1 & \text{if } i = r \text{ and } i \neq j, \\ 0 & \text{otherwise,} \end{cases} \quad (2 \le r \le k)$$

we can see that

$$\rho \ge \frac{\frac{1}{\beta(\mathbf{D})} + (k-1)(\frac{1}{\beta(\mathbf{D})} - 1)}{\frac{1}{\beta(\mathbf{D})}} = k - (k-1)\beta(\mathbf{D}).$$

References

[1] P. Barcaccia, M.A. Bounccelli, and M.D. Ianni. Complexity of Minimum Length Scheduling for Precedence Constrained Messages in Distributed Systems. *IEEE Transactions on Parallel and Distributed Systems*, 11(10):1090–1102, 2000.