Explicit Construction of Optimal Fault-Tolerant Linear Arrays *

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1 Introduction

We consider the following problem motivated by the design of fault-tolerant linear array multiprocessor systems. Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. $\Delta(G)$ is the maximum degree of a vertex in G. For any $S \subseteq V(G)$, $G - S$ is the graph obtained from G by deleting the vertices of S together with the edges incident with the vertices in S. Let k be a positive integer. A graph G is called a k-FT (k-fault-tolerant) graph for a graph H if $G - F$ contains H as a subgraph for every $F \subseteq V(G)$ with $|F| \le k$. Our problem is to construct a k-FT graph G for an n-vertex path P_n such that both $|V(G)|$ and $\Delta(G)$ are as small as possible.

A large amount of research has been devoted to constructing k-FT graphs for P_n [1–3, 6–8, 10–13]. Among others, Bruck, Cypher, and Ho [2] show a k-FT graph for P_n with $n + k^2$ vertices and maximum degree of 4. Zhang [12, 13] shows a k-FT graph for P_n with $n + O(k \log k)$ vertices and $O(\log k)$ maximum degree, and a k-FT graph for P_n with $n + O(k \log^2 k)$ vertices and $O(1)$ maximum degree. Zhang [12, 13] also raised the following open question: Is it possible to construct an explicit k-FT graph for P_n with $n + O(k)$ vertices and $O(1)$ maximum degree? It should be noted that such a k-FT graph is optimal in the sense that every k-FT graph for P_n has $n + \Omega(k)$ vertices and $\Omega(1)$ maximum degree.

In this paper, we settle the question by showing the following.

Theorem 1 *For any positive integers n and k, we can explicitly construct a k-FT graph G for P_n such that $|V(G)| = n + O(k)$ and $\Delta(G) = 3$.* ∎

We note that Alon and Chung [1] proved that for any positive integers n and $k = \Omega(n)$, we can explicitly construct a k-FT graph G for P_n such that $|V(G)| = n + O(k)$ and $\Delta(G) = O(1)$.

Due to space limitation, we omit the proofs of Lemmas 1–4 below.

2 Proof of Theorem 1

Let $\Gamma_G(v)$ denote the set of vertices adjacent to v in a graph G, $\Gamma_G(X) = \bigcup_{v \in X} \Gamma_G(v)$, and $\partial X = \Gamma_G(X) - X$ for any $X \subseteq V(G)$. We define that $\deg_G(v) = |\Gamma_G(v)|$, and $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$.

In order to prove Theorem 1, we first need a few results on magnifiers.

2.1 Magnifiers

Let $c \le 1$. A graph G is an (n, d, c)-magnifier if the following three conditions are satisfied:

1. $|V(G)| = n$;
2. $\Delta(G) \le d$;
3. $|\partial X| \ge c|X|$ for every $X \subset V(G)$ with $|X| \le n/2$.

For any positive integer m, let $[m] = \{0, 1, \dots, m-1\}$. For any positive integer m, $M(m)$ is the graph defined as follows: $V(M(m)) = [m]^2$; Each vertex $[i, j] \in V(M(m))$ is connected with 12 vertices $[i \pm 2j, j], [i \pm (2j+1), j], [i \pm (2j+2)], [i, j \pm 2i], [i, j \pm (2i+1)], [i, j \pm (2j+2)]$, each by an edge. Lemma 1 is immediate from a result on expanders in [5].

Lemma 1 *For any positive integer m, $M(m)$ is an $(m^2, 12, (2-\sqrt{3})/4)$-magnifier.* ∎

Lemma 2 *If G is an (n, d, c)-magnifier and $k \le cn/4$ is a positive integer then $G - F$ contains a connected component of size at least $n - (1 + 1/c)k$ for any $F \subset V(G)$ with $|F| \le k$.* ∎

2.2 Products of Magnifiers and Paths

For any two graphs G and H, the product of G and H, denoted by $G \times H$, is the graph defined as follows: $V(G \times H) = V(G) \times V(H)$; Any two vertices $[u, x]$ and $[v, y]$ in $G \times H$ are joined by an edge if one of the following conditions is satisfied:

1. $(u, v) \in E(G)$ and $x = y$, or
2. $u = v$ and $(x, y) \in E(H)$.

Lemma 3 *Let n_1 and n_2 be two positive integers, and k be a positive integer with $k \le \min\{n_1/4, n_2 - 1\}$. If G is an (n_1, d, c)-magnifier for some positive integer d and positive number c then $G \times P_{n_2} - F$ contains a connected component of size at least $n - (1 + 1/c)k$ for any $F \subseteq V(G \times P_{n_2})$ with $|F| = k$, where $n = n_1 n_2$ is the number of vertices in $G \times P_{n_2}$.* ∎

2.3 Proof of Theorem 1

Let $c = (2 - \sqrt{3})/4$. we define that

$$H_{n,k} = \begin{cases} M(m_1) \times P_{n_2} & \text{if } 1 \le k \le \sqrt{n/8}, \\ M(m_2) & \text{if } \sqrt{n/8} < k \le cn/(3-c), \\ H_{n_3,k} & \text{otherwise,} \end{cases}$$

where m_1, n_2, m_2, n_3 are integers such that $(m_1 - 1)^2 < 4k \le m_1^2$, $n_2 = \lceil (n + (1 + 1/c)k)/m_1^2 \rceil$, $(m_2 - 1)^2 < n + (1 + 1/c)k \le m_2^2$, and $n_3 = \lceil (3 - c)k/c \rceil$. By Lemmas 1, 2, and 3, we obtain the following lemma.

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Lemma 4 *$H_{n,k}$ satisfies the following three conditions:*

(c1) *$H_{n,k} - F$ contains a connected component of size at least n for any $F \subseteq V(H_{n,k})$ with $|F| \leq k$,*

(c2) *$|V(H_{n,k})| \leq n + \gamma k + \delta$ for some constants γ and δ, and*

(c3) $\Delta(H_{n,k}) \leq 14$. ∎

Now, we are ready to prove Theorem 1.

Proof of Theorem 1: Let $d = 14$, $n' = \lceil n/2d \rceil$, and f_u be a one-to-one mapping from $\Gamma_{H_{n',k}}(u)$ to $[d]$. $G_{n,k}$ is the graph defined as follows: $V(G_{n,k}) = V(H_{n',k}) \times [2d]$; Any two vertices $[u,i],[v,j] \in V(G_{n,k})$ are connected by an edge if one of the following two conditions is satisfied:

(i) $u = v$ and $j = (i \pm 1) \bmod (2d)$;

(ii) $(u,v) \in E(H_{n',k})$, $i = 2f_u(v) + r$, $j = 2f_v(u) + r$, and $r \in [2]$.

We are going to show that $G_{n,k}$ is a desired k-FT graph for P_n. It is easy to see the following two lemmas.

Lemma 5 $|V(G_{n,k})| \leq n + 2d\gamma k + 2d(\delta + 1)$. ∎

Lemma 6 $\Delta(G_{n,k}) = 3$. ∎

It remains to show the following:

Lemma 7 *$G_{n,k}$ is a k-FT graph for P_n.*

Proof : We show that for any $F \subseteq V(G_{n,k})$ with $|F| \leq k$, $G_{n,k} - F$ contains P_n as a subgraph. Let $F' = \{v \in V(H_{n',k}) : [v,j] \in F, j \in [2d]\}$. Since $|F'| \leq |F| \leq k$ by definition, $H_{n',k} - F'$ contains a connected component $\mathcal{H}$ of size at least n'. Let T denote a spanning tree of $\mathcal{H}$. A vertex r of T is designated as a root, and T is considered as a rooted tree. For any $v \in V(T)$, let $T(v)$ is a subtree of T consisting of the descendants of v. Define that

$$\begin{aligned} X(v) &= \{[v,j] : j \in [2d]\}, \\ Y(v) &= \{[u,i] : u \in T(v), i \in [2d]\}, \end{aligned}$$

and $\mathcal{G}(v)$ denote the subgraph of $G_{n,k}$ induced by $Y(v)$.

Claim 1 *Let $v_0, \ldots, v_{m-1}$ be the children of $u \in V(T)$. If $\mathcal{G}(v_l)$ has a Hamilton cycle for every $l \in [m]$ then $\mathcal{G}(u)$ has a Hamilton cycle.*

Proof of Claim 1: For each $l \in [m]$, let C^l denote a Hamilton cycle of $\mathcal{G}(v_l)$, and let $C(u)$ denote the subgraph of $G_{n,k}$ induced by $X(u)$, which is isomorphic to C_{2d}. Define C as the graph obtained from $C^0, C^1, \ldots, C^{m-1}$, and $C(u)$ by replacing two edges $([u, 2f_u(v_l)], [u, 2f_u(v_l)+1])$ and $([v_l, 2f_{v_l}(u)], [v_l, 2f_{v_l}(u) + 1])$ with $([u, 2f_u(v_l)], [v_l, 2f_{v_l}(u)])$ and $([u, 2f_u(v_l) + 1], [v_l, 2f_{v_l}(u) + 1])$ for each $l \in [m]$. It is easy to see that C is a Hamilton cycle of $\mathcal{G}(u)$. ∎

It is easy to see that $\mathcal{G}(v)$ has a Hamilton cycle if $v \in V(T)$ is a leaf. Hence, we obtain by Claim 1 a Hamilton cycle of $\mathcal{G}(r)$. Since

$$|V(\mathcal{G}(r))| = 2d \cdot |V(T)| \geq 2dn_1 \geq 2d \cdot \frac{n}{2d} = n,$$

$G_{n,k} - F$ contains P_n as a subgraph. Hence, we conclude that $G_{n,k}$ is a k-FT graph for P_n. ∎

Lemmas 5, 6, and 7 complete the proof of Theorem 1. ∎

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