A Note on the Implementation of de Bruijn Networks by the Optical Transpose Interconnection System *

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1 Introduction

This note shows an efficient implementation of de Bruijn networks by the Optical Transpose Interconnection System (OTIS).

The OTIS architecture was proposed by Marsden, Manchand, and Esener[1] to implement networks by free space optical interconnections. This architecture consists of the input and output arrays, and a pair of lenslet arrays between I/O arrays. The input array consists of p groups of q transmitters, and the output array consists of q groups of p receivers. The lenslet arrays consists of $p+q$ lenses. This architecture connects transmitter (i,j) to receiver $(q-j-1, p-i-1)$, $0 \le i \le p-1, 0 \le j \le q-1$.

The OITS architecture is represented by a bipartite digraph $OTIS(p,q)$ defined as follows. Let $V(G)$ and $A(G)$ denote the vertex set and edge set of a digraph G, respectively. An arc from a vertex u to v is denoted by $[u,v]$. We define that $Z_n = \{0, 1, \ldots, n-1\}$ for any positive integer n. $OTIS(p,q)$ is a bipartite digraph defined as:

$$V(OTIS(p,q)) = \{(i,j)_t | i \in Z_p, j \in Z_q\} \cup \{(k,l)_r | k \in Z_q, l \in Z_p\};$$

$$A(OTIS(p,q)) = \{[(i,j)_t, (k,l)_r] | i+l = p-1, j+k = q-1\}.$$

Let $H(p,q,d)$ be a d-regular digraph obtained from $OTIS(p,q)$ as follows:

$$V(H(p,q,d)) = Z_{\lfloor pq/d \rfloor};$$

$$A(H(p,q,d)) = \left\{ [u,v] \;\middle|\; \begin{array}{l} [(i,j)_t, (k,l)_r] \in A(OTIS(p,q)), \\ u = \lfloor (iq+j)/d \rfloor, v = \lfloor (kp+l)/d \rfloor \end{array} \right\}.$$

An OTIS architecture with $2+8$ lenses shown in Fig.1 is represented by $OTIS(2,8)$ shown in Fig.2. $H(2,8,2)$ shown in Fig.3 is a 2-regular digraph obtained from $OTIS(2,8)$.

$OTIS(p,q)$ is called an OTIS layout for a d-regular digraph G if $H(p,q,d)$ is isomorphic to G. Our problem is to find an OTIS layout for a given digraph such that the number of lenses $p+q$ is as small as possible.

The de Bruijn network has been extensively studied in connection with parallel and distributed computing. For any integer $d \ge 2$, the d-ary de Bruijn network of dimension D, denoted by $B(d,D)$, is a d-regular digraph defined as follows:

$$V(B(d,D)) = Z_d^D;$$

$$A(B(d,D)) = \left\{ [u,v] \;\middle|\; \begin{array}{l} u, v \in V(B(d,D)) \\ u_i = v_{i+1} \quad \text{for} \quad \forall i \in Z_{D-1} \end{array} \right\},$$

where a vertex $x \in V(B(d,D))$ is denoted by $(x_0, x_1, \ldots, x_{D-1})$. $B(2,3)$ is shown in Fig.4. Since $B(2,3)$ is isomorphic to $H(2,8,2)$, $OTIS(2,8)$ is an OTIS layout for $B(2,3)$.

Coudert, Ferreira, and Perennes[2] show that if D is even, $B(d,D)$ has an efficient OTIS layout with $O(d\sqrt{N})$ lenses, where $N = d^D = |V(B(d,D))|$. It should be noted that the number of lenses of an OTIS layout for an N-vertex d-regular digraph is $\Omega(\sqrt{dN})$ as can be easily seen[2].

We show a natural extension to the result above by proving that for any positive integer D, $B(d,D)$ has an efficient OTIS layout. The number of lenses of the OTIS layout is $O(d^{\frac{5}{2}}\sqrt{N})$ if $D \equiv 1 \pmod 4$, $O(d^{\frac{3}{2}}\sqrt{N})$ if $D \equiv 3 \pmod 4$, and $O(d\sqrt{N})$ if D is even.

2 Main Results

Theorem 1 *For any positive integers p' and q', $H(d^{p'}, d^{q'}, d)$ is isomorphic to $B(d, p'+q'-1)$ if and only if p' and q' are relatively prime.*

Proof: We denote $H(d^{p'}, d^{q'}, d)$ by H, and $p'+q'-1$ by D. It is easy to see that $H(d^{p'}, d^{q'}, d)$ is not isomorphic to $B(d, p'+q'-1)$ if $p'q' = 0$. So we assume that $p'q' \ne 0$. A permutation on Z_n is a bijective mapping on Z_n. We define a permutation f_D on Z_D as follows:

$$f_D(i) = \begin{cases} i+p' & \text{if } i < q'-1, \\ p'-1 & \text{if } i = q'-1, \\ (i+p'-1 \bmod D) & \text{otherwise.} \end{cases}$$

where, $i \in Z_D$. For a permutation f on Z_n, the repeated composition of f with itself is defined inductively as follows: 1) f^0 is the identity permutation;

*OTISによる de Bruijn ネットワークの実装について
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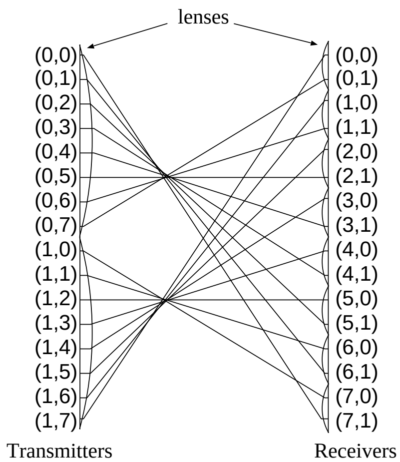

Figure 1: OTIS architecture with 2 + 8 lenses

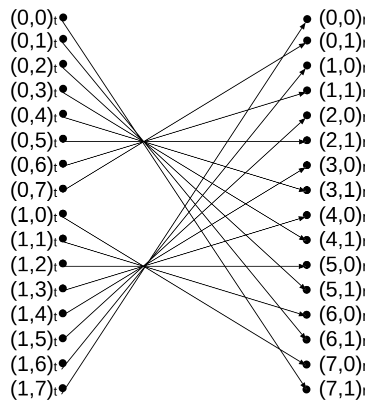

Figure 2: $OTIS(2, 8)$

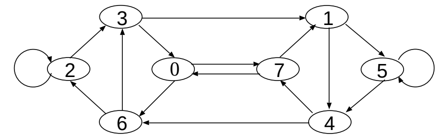

Figure 3: $H(2, 8, 2)$

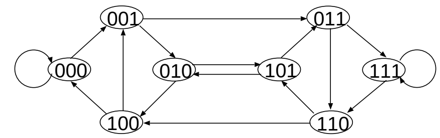

Figure 4: $B(2, 3)$

2) $f^{i+1} = f \circ f^i$. f is said to be cyclic if $f^j(i) \neq i$ for any $i \in Z_n$ and $j \in Z_{n-1}$. The following lemma is proved in [2].

Lemma I [2] For any positive integers p' and q', $H(d^{p'}, d^{q'}, d)$ is isomorphic to $B(d, D)$ if and only if f_D is cyclic.

Lemma 1 *f_D is cyclic if and only if p' and q' are relatively prime.*

Proof of Lemma 1: We define a permutation g_{D+1} on Z_{D+1} as follows:

$$g_{D+1}(i) = (i + p' \bmod (D+1)).$$

It is easy to see that

$$g_{D+1}(i) = \begin{cases} D & \text{if } i = q' - 1, \\ f_D(q'-1) & \text{if } i = D, \\ f_D(i) & \text{otherwise.} \end{cases}$$

It follows that f_D is cyclic if and only if g_{D+1} is cyclic. By the definition of g_{D+1}, g_{D+1} is cyclic if and only if p' and $D + 1 = p' + q'$ are relatively prime. Thus, f_D is cyclic if and only if p' and q' are relatively prime.

From Lemmas I and 1, we obtain the theorem.

Theorem 2 *For any positive integers D and $d \geq 2$, $B(d, D)$ has an OTIS layout. The number of lenses of the OTIS layout is $O(d^{\frac{5}{2}}\sqrt{N})$ if $D \equiv 1 \pmod 4$, $O(d^{\frac{3}{2}}\sqrt{N})$ if $D \equiv 3 \pmod 4$, and $O(d\sqrt{N})$ if D is even, where $N = d^D = |V(B(d, D))|$.*

Proof: The theorem follows from Theorem 1 if we choose p' and q' as follows:

$$(p', q') = \begin{cases} (m, m+1) & \text{if } D = 2m, \\ (2m'-1, 2m'+1) & \text{if } D = 4m'-1, \\ (2m'-1, 2m'+3) & \text{if } D = 4m'+1, \\ (1, 1) & \text{if } D = 1. \end{cases}$$

where, m and m' are positive integers.

3 Concluding Remarks

- The number of lenses of our OTIS layout is optimal if d is fixed. Closing the gap between the upper and lower bounds of the number of lenses is an open problem.
- It should be noted that not all the digraphs have the OTIS layout. It is an interesting open problem to characterize the digraphs that have OTIS layouts.

References

[1] G.C. Marsden, P.J.Marchand, P.Harvey, and S.C.Esener "Optical transpose interconnection system architectures", Optics Letters, Vol.18, No.13, 1083-1085, 1993

[2] D.Coudert, A.Ferreira and S.Perennes. "Isomorphisms of the de Bruijn and Free Space Optical Networks", Networks, Vol.40(3),155-164, 2002