3Q-08

Sparse Networks Tolerating Random Faults for Tree-Like and Butterfly-Like Networks *

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1 Introduction

This paper considers the following problem in connection with the design of fault-tolerant interconnection networks for multiprocessor systems: Given an N-vertex graph G, construct an $O(N)$-vertex graph G^* with a minimum number of edges such that even after deleting each vertex from G^* independently with constant probability, the remaining graph contains G as a subgraph, with probability converging to 1, as $N \to \infty$. G^* is called an RFT(random-fault-tolerant) graph for G. Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G, respectively. Fraigniaud, Kenyon, and Pelc showed in [4] that for any N-vertex graph G, there exists an RFT graph for G with $O(|E(G)|\log^2 N)$ edges, and that for any N-vertex graph G with $O(N)$ edges and maximum degree of $\Omega(N)$, any RFT graph for G has $\omega(|E(G)|)$ edges. It is an interesting open problem posed in [4] to decide whether any N-vertex graph G has an RFT graph with $O(|E(G)|\log N)$ edges. It is known that if G is a path[1], cycle[4], or tree with bounded vertex degree[3], we can construct an RFT graph for G with $O(|E(G)|)$ edges ; if G is an N-vertex mesh or torus[5], we can construct an RFT graph for G with $O(|E(G)|\log\log N)$ edges; if G is an N-vertex tree[4], circulant graph, hypercube, de Bruijn graph, shuffle-exchange graph, or cube-connected-cycles[6], we can construct an RFT graph for G with $O(|E(G)|\log N)$ edges.

This paper shows that if G is an N-vertex partial k-tree, butterfly, wrapped butterfly, or Beneš graph, we can construct an RFT graph for G with $O(|E(G)|\log N)$ edges. The open problem mentioned above remains unresolved.

2 A General Construction of RFT Graphs

In this section, we review a general method to construct RFT graphs proposed in [6]. For any positive integer h, let $[h] = \{0, 1, \ldots, h-1\}$. A collection $\{S_0, S_1, \cdots, S_{h-1}\}$ of subsets of S is called a partition of S if $\bigcup_{i\in[h]} S_i = S$ and $S_i \cap S_j = \emptyset$ for any $i \neq j$. For an N-vertex graph G and a partition $\mathcal{V} = \{V_0, V_1, \ldots, V_{h-1}\}$ of $V(G)$, define that $\Lambda(G,\mathcal{V}) = \{(i,j) | \exists(u,v) \in E(G)(u \in V_i, v \in V_j)\}$ and $\lambda(G,\mathcal{V}) = |\Lambda(G,\mathcal{V})|$. Let $0 < p < 1$ be the probability for each vertex to be deleted.

Suppose that $\mathcal{V} = \{V_0, V_1, \ldots, V_{h-1}\}$ is a partition of $V(G)$ such that $|V_i| \le \alpha \ln N$ for any $i \in [h]$ and $h \le \beta N/\ln N$ for some fixed positive numbers α and β . Let $V_0^*, V_1^*, \ldots, V_{h-1}^*$ be h sets such that $|V_i^*| = \lceil \gamma \ln N \rceil$ for any $i \in [h]$ and $V_i^* \cap V_j^* = \emptyset$ for any $i \neq j$, where $\gamma = (\sqrt{2\alpha+1}+1)^2/2(1-p)$. $G^*[\mathcal{V}]$ is the graph defined as follows:

$$V(G^*[\mathcal{V}]) = V_0^* \cup V_1^* \cup \cdots \cup V_{h-1}^*$$

$$E(G^*[\mathcal{V}]) = \left\{ (u^*, v^*) \;\middle|\; \begin{array}{c} u^* \in V_i^*, v^* \in V_j^* \\ (i,j) \in \Lambda(G,\mathcal{V}) \end{array} \right\}$$

The following theorem is proved in [6].

Theorem I *$G^*[\mathcal{V}]$ is an RFT graph for G with $O(\lambda(G,\mathcal{V})\log^2 N)$ edges. In particular, if $\lambda(G,\mathcal{V}) = O(|E(G)|/\log N)$ then $G^*[\mathcal{V}]$ is an RFT graph with $O(|E(G)|\log N)$ edges.* ∎

3 RFT Graphs for Partial k-Trees

3.1 Partial k-Trees

A tree decomposition of a graph G is a pair $(T, \mathcal{X})$, where T is a tree and $\mathcal{X} = \{X_t \subseteq V(G) | t \in V(T)\}$ is a family of subsets of $V(G)$, satisfying the following three conditions:

1. $V(G) = \bigcup_{t\in V(T)} X_t$;
2. for every $(u,v) \in E(G)$, there exists $t \in V(T)$ such that $u, v \in X_t$;
3. for every $r, s, t \in V(T)$, if s is on the path between r and t in T then $X_r \cap X_t \subseteq X_s$.

The width of $(T,\mathcal{X})$ is $\max\{|X_t| - 1 | t \in V(T)\}$. The treewidth of G is the minimum width over all possible tree decompositions of G.

A graph of treewidth at most k is called a partial k-tree. It is easy to see that a tree is a partial 1-tree.

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3.2 RFT Graphs

We assume in this section that k is a fixed positive integer. Let G be a connected partial k-tree with N vertices, and let $(T, \mathcal{X})$ be a tree decomposition of G with width at most k, where $\mathcal{X} = \{X_t \subseteq V(G) \mid t \in V(T)\}$, and T is considered as a rooted tree with root r. It is not difficult to see that $|E(G)| \leq kN$.

Lemma 1 *There exists a partition* $\mathcal{Y} = \{Y_0, Y_1, \ldots, Y_{l-1}\}$ *of* $V(T)$ *that satisfies the following four conditions:*

1. $l = O(N/\log N)$;
2. *For any* $i \in [l-1]$, *there exists a vertex* $t_i \in V(T)$ *such that the parent of each vertex of* Y_i *is contained in* $Y_i \cup \{t_i\}$;
3. $r \in Y_{l-1}$, *and the parent of each vertex of* $Y_{l-1} - \{r\}$ *is contained in* Y_{l-1}.
4. $\mathcal{V} = \{\bigcup_{t \in Y_i} X_t - X_{t_i} \mid i \in [l-1]\} \cup \{\bigcup_{t \in Y_{l-1}} X_t\}$ *is a partition of* $V(G)$ *such that the size of each block is* $O(\log N)$.

Theorem 1 *A partial* k*-tree* G *with* N *vertices has an RFT graph with* $O(|E(G)| \log N)$ *edges.*

Proof : (Sketch) We can prove that $\lambda(G, \mathcal{V}) = O(|E(G)|/\log N)$ for partition $\mathcal{V}$ defined in Lemma 1. Thus we have the theorem from Theorem I. ∎

It should be noted that Theorem 1 is a natural generalization of a result for trees shown in [4], since trees are partial 1-trees.

4 RFT Graphs for Butterfly-Like Graphs

4.1 Butterfly-Like Graphs

For any $v = [v_1, v_2, \ldots, v_n] \in [2]^n$, let $\chi_i(v) = [v_1, v_2, \ldots, v_{i-1}, \bar{v_i}, v_{i+1}, \ldots, v_n]$ and $\rho_i(v) = [v_1, v_2, \ldots, v_i]$, where $\bar{v_i}$ denotes the complement of v_i, that is $\bar{v_i} = 1$ if $v_i = 0$, and $\bar{v_i} = 0$ otherwise. The n-dimensional butterfly $B(n)$ is the graph defined as follows: $V(B(n)) = [2]^n \times [n+1]$; $E(B(n)) = \{([u,i],[v,i+1]) | v = u \text{ or } v = \chi_{i+1}(u)\}$, where $u, v \in [2]^n$ and $i \in [n]$. It is easy to see that $|V(B(n))| = N = (n+1)2^n$, and $|E(B(n))| \leq 2N$.

The n-dimensional wrapped butterfly is the graph obtained from $B(n)$ by merging vertices $[v, 0]$ and $[v, n]$ for each $v \in [2]^n$. The n-demensional wrapped butterfly has $n2^n$ vertices, each of degree 4. The Beneš graph consists of back-to-back butterflies. The n-dimensional Beneš graph has $(2n+1)2^n$ vertices.

4.2 RFT Graphs

Let $V_{[x,i]} = \{[u,i] \in V(B(n)) \mid \rho_{n-\lceil \log n \rceil}(u) = x\}$ for any $x \in [2]^{n-\lceil \log n \rceil}$ and $i \in [n+1]$, and let $\mathcal{V} = \{V_{[x,i]} \mid x \in [2]^{n-\lceil \log n \rceil}, i \in [n+1]\}$. It is easy to see that $\mathcal{V}$ is a partition of $V(B(n))$ such that $|V_{[x,i]}| = O(\log N)$ for any $x \in [2]^{n-\lceil \log n \rceil}$ and $i \in [n+1]$, and $|\mathcal{V}| = O(N/\log N)$.

Theorem 2 *An* N*-vertex butterfly* $B(n)$ *has an RFT graph with* $O(|E(B(n))| \log N)$ *edges.*

Proof : (Sketch) We can prove that $\lambda(B(n), \mathcal{V}) = O(|E(G)|/\log N)$ for partition $\mathcal{V}$ of $V(B(n))$ defined above. Thus we have the theorem from Theorem I. ∎

Similar argument can be applied to wrapped butterflies and Beneš graphs.

Theorem 3 *If* G *is a wrapped butterfly or Beneš graph with* N *vertices,* G *has an RFT graph with* $O(|E(G)| \log N)$ *edges.*

Notice that Theorems 2 and 3 together with results in [6] cover the well-known classes of hypercubic graphs.

Acknowledgments: The authors are grateful to Prof. Y. Kajitani for his encouragement. The research is a part of CAD21 Project at TIT.

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