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# Balanced trefoil decomposition algorithm of complete tripartite multi-graphs

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## 1. Introduction

Let $K_{n_1,n_2,n_3}$ denote *the complete tripartite graph* with partite sets $V_1$, $V_2$, $V_3$ of $n_1$, $n_2$, $n_3$ vertices each. *The complete tripartite multi-graph* $\lambda K_{n_1,n_2,n_3}$ is the complete tripartite graph $K_{n_1,n_2,n_3}$ in which every edge is taken $\lambda$ times. *The trefoil* (or *the* 3-*windmill*) is a graph of 3 edge-disjoint triangles with a common vertex and the common vertex is called *the center of the trefoil.* When $\lambda K_{n_1,n_2,n_3}$ is decomposed into edge-disjoint sum of trefoils, it is called that $\lambda K_{n_1,n_2,n_3}$ *has a trefoil decomposition.* Moreover, when every vertex of $\lambda K_{n_1,n_2,n_3}$ appears in the same number of trefoils, it is called that $\lambda K_{n_1,n_2,n_3}$ *has a balanced trefoil decomposition* and this number is called *the replication number.*

## 2. Balanced trefoil decomposition of $\lambda K_{n_1,n_2,n_3}$

**Notation.** We denote a trefoil passing through $v_1 - v_2 - v_3 - v_1 - v_4 - v_5 - v_1 - v_6 - v_7 - v_1$ by $\{(v_1, v_2, v_3), (v_1, v_4, v_5), (v_1, v_6, v_7)\}$.

**Lemma 1.** If $\lambda K_{n,n,n}$ has a balanced trefoil decomposition, then $s\lambda K_{n,n,n}$ has a balanced trefoil decomposition.

**Lemma 2.** If $\lambda K_{n,n,n}$ has a balanced trefoil decomposition, then $\lambda K_{sn,sn,sn}$ has a balanced trefoil decomposition.

**Theorem 1.** $\lambda K_{n_1,n_2,n_3}$ has a balanced trefoil decomposition if and only if $\lambda n_1 = \lambda n_2 = \lambda n_3 \equiv 0 \pmod 9$, $n_1 \geq 3$.

**Proof. (Necessity)** Suppose that $\lambda K_{n_1,n_2,n_3}$ has a balanced trefoil decomposition. Let $b$ be the number of trefoils and $r$ be the replication number. Then $b = \lambda(n_1n_2 + n_1n_3 + n_2n_3)/9$ and $r = 7\lambda(n_1n_2 + n_1n_3 + n_2n_3)/9(n_1 + n_2 + n_3)$. Among $r$ trefoils having vertex $v$ in $V_i$, ler $r_{ij}$ be the number of trefoils in which the centers are in $V_j$. Then $r_{11} + r_{12} + r_{13} = r_{21} + r_{22} + r_{23} = r_{31} + r_{32} + r_{33} = r$. Counting the number of vertices adjacent to vertex $v$ in $V_1$, $3r_{11} + r_{12} + r_{13} = \lambda n_2$ and $3r_{11} + r_{12} + r_{13} = \lambda n_3$. Counting the number of vertices adjacent to vertex $v$ in $V_2$, $r_{21} + 3r_{22} + r_{23} = \lambda n_1$ and $r_{21} + 3r_{22} + r_{23} = \lambda n_3$. Counting the number of vertices adjacent to vertex $v$ in $V_3$, $r_{31} + r_{32} + 3r_{33} = \lambda n_1$ and $r_{31} + r_{32} + 3r_{33} = \lambda n_2$. Therefore, $n_1 = n_2 = n_3$.
Put $n_1 = n_2 = n_3 = n$. Then $b = \lambda n^2/3$, $r = 7\lambda n/9$, $r_{11} = r_{22} = r_{33} = \lambda n/9$ and $r_{12} + r_{13} = r_{21} + r_{23} = r_{31} + r_{32} = 2\lambda n/3$. Thus $\lambda n \equiv 0 \pmod 9$. Since a trefoil is a subgraph of $\lambda K_{n,n,n}$, $n \geq 3$.

**(Sufficiency) Case 1.** $n \equiv 0$ **(mod 9).** Put $n = 9s$. When $s = 1$, let $V_1 = \{1, 2, ..., 9\}$, $V_2 = \{10, 11, ..., 18\}$, $V_3 = \{19, 20, ..., 27\}$.
Construct a balanced trefoil decomposition of $K_{9,9,9}$:
$B_1 = \{(1, 10, 19), (1, 11, 20), (1, 12, 21)\}$
$B_2 = \{(2, 10, 20), (2, 11, 21), (2, 12, 19)\}$

$B_3 = \{(3,10,21),(3,11,19),(3,12,20)\}$
$B_4 = \{(4,13,22),(4,14,23),(4,15,24)\}$
$B_5 = \{(5,13,23),(5,14,24),(5,15,22)\}$
$B_6 = \{(6,13,24),(6,14,22),(6,15,23)\}$
$B_7 = \{(7,16,25),(7,17,26),(7,18,27)\}$
$B_8 = \{(8,16,26),(8,17,27),(8,18,25)\}$
$B_9 = \{(9,16,27),(9,17,25),(9,18,26)\}$
$B_{10} = \{(10,22,7),(10,23,8),(10,24,9)\}$
$B_{11} = \{(11,22,8),(11,23,9),(11,24,7)\}$
$B_{12} = \{(12,22,9),(12,23,7),(12,24,8)\}$
$B_{13} = \{(13,25,1),(13,26,2),(13,27,3)\}$
$B_{14} = \{(14,25,2),(14,26,3),(14,27,1)\}$
$B_{15} = \{(15,25,3),(15,26,1),(15,27,2)\}$
$B_{16} = \{(16,19,4),(16,20,5),(16,21,6)\}$
$B_{17} = \{(17,19,5),(17,20,6),(17,21,4)\}$
$B_{18} = \{(18,19,6),(18,20,4),(18,21,5)\}$
$B_{19} = \{(19,7,13),(19,8,14),(19,9,15)\}$
$B_{20} = \{(20,7,14),(20,8,15),(20,9,13)\}$
$B_{21} = \{(21,7,15),(21,8,13),(21,9,14)\}$
$B_{22} = \{(22,1,16),(22,2,17),(22,3,18)\}$
$B_{23} = \{(23,1,17),(23,2,18),(23,3,16)\}$
$B_{24} = \{(24,1,18),(24,2,16),(24,3,17)\}$
$B_{25} = \{(25,4,10),(25,5,11),(25,6,12)\}$
$B_{26} = \{(26,4,11),(26,5,12),(26,6,10)\}$
$B_{27} = \{(27,4,12),(27,5,10),(27,6,11)\}$.
Therefore, $\lambda K_{n,n,n}$ has a balanced trefoil decomposition.
**Case 2. $n \equiv 0 \pmod 3$ and $\lambda \equiv 0 \pmod 3$.** Put $n = 3s$. When $s = 1$, let $V_1 = \{1,2,3\}$, $V_2 = \{4,5,6\}$, $V_3 = \{7,8,9\}$.
Construct a balanced trefoil decomposition of $3K_{3,3,3}$:
$B_1 = \{(1,4,7),(1,5,8),(1,6,9)\}$
$B_2 = \{(2,4,8),(2,5,9),(2,6,7)\}$
$B_3 = \{(3,4,9),(3,5,7),(3,6,8)\}$
$B_4 = \{(4,7,1),(4,8,2),(4,9,3)\}$
$B_5 = \{(5,7,2),(5,8,3),(5,9,1)\}$
$B_6 = \{(6,7,3),(6,8,1),(6,9,2)\}$
$B_7 = \{(7,1,4),(7,2,5),(7,3,6)\}$
$B_8 = \{(8,1,5),(8,2,6),(8,3,4)\}$
$B_9 = \{(9,1,6),(9,2,4),(9,3,5)\}$.
Therefore, $\lambda K_{n,n,n}$ has a balanced trefoil decomposition.
**Case 3. $n \geq 3$ and $\lambda \equiv 0 \pmod 9$.** Let $V_1 = \{1,2,...,n\}$, $V_2 = \{1',2',...,n'\}$, $V_3 = \{1'',2'',...,n''\}$.
Construct a balanced trefoil decomposition of $9K_{n,n,n}$:
$B_{ij}^{(1)} = \{(i,j',(i+j-1)''),(i,(j+1)',(i+j)''),(i,(j+2)',(i+j+1)'')\}$
$B_{ij}^{(2)} = \{(i',j'',i+j-1),(i',(j+1)'',i+j),(i',(j+2)'',i+j+1)\}$
$B_{ij}^{(3)} = \{(i'',j,(i+j-1)'),(i'',j+1,(i+j)'),(i'',j+2,(i+j+1)')\}$.
Therefore, $\lambda K_{n,n,n}$ has a balanced trefoil decomposition.

## References

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[2] W. D. Wallis, Combinatorial Designs. Marcel Dekker, New York and Basel (1988).