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Balanced bowtie decomposition algorithm of complete tripartite multi-graphs

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1. Introduction

Let K_{n_1,n_2,n_3} denote *the complete tripartite graph* with partite sets V_1, V_2, V_3 of n_1, n_2, n_3 vertices each. *The complete tripartite multi-graph* $\lambda K_{n_1,n_2,n_3}$ is the complete tripartite graph K_{n_1,n_2,n_3} in which every edge is taken λ times. *The bowtie* (or *the 2-windmill*) is a graph of 2 edge-disjoint triangles with a common vertex and the common vertex is called *the center of the bowtie.* When $\lambda K_{n_1,n_2,n_3}$ is decomposed into edge-disjoint sum of bowties, it is called that $\lambda K_{n_1,n_2,n_3}$ *has a bowtie decomposition.* Moreover, when every vertex of $\lambda K_{n_1,n_2,n_3}$ appears in the same number of bowties, it is called that $\lambda K_{n_1,n_2,n_3}$ *has a balanced bowtie decomposition* and this number is called *the replication number.*

2. Balanced bowtie decomposition of $\lambda K_{n_1,n_2,n_3}$

Notation. We denote a bowtie passing through $v_1-v_2-v_3-v_1-v_4-v_5-v_1$ by $\{(v_1,v_2,v_3),(v_1,v_4,v_5)\}$.

Lemma 1. If $\lambda K_{n,n,n}$ has a balanced bowtie decomposition, then $s\lambda K_{n,n,n}$ has a balanced bowtie decomposition.

Lemma 2. If $\lambda K_{n,n,n}$ has a balanced bowtie decomposition, then $\lambda K_{sn,sn,sn}$ has a balanced bowtie decomposition.

Theorem 1. $\lambda K_{n_1,n_2,n_3}$ has a balanced bowtie decomposition if and only if $\lambda n_1 = \lambda n_2 = \lambda n_3 \equiv 0 \pmod 6$, $n_1 \geq 2$.

Proof. **(Necessity)** Suppose that $\lambda K_{n_1,n_2,n_3}$ has a balanced bowtie decomposition. Let b be the number of bowties and r be the replication number. Then $b = \lambda(n_1n_2+n_1n_3+n_2n_3)/6$ and $r = 5\lambda(n_1n_2+n_1n_3+n_2n_3)/6(n_1+n_2+n_3)$. Among r bowties having vertex v in V_i, ler r_{ij} be the number of bowties in which the centers are in V_j. Then $r_{11}+r_{12}+r_{13} = r_{21}+r_{22}+r_{23} = r_{31}+r_{32}+r_{33} = r$. Counting the number of vertices adjacent to vertex v in V_1, $2r_{11}+r_{12}+r_{13} = \lambda n_2$ and $2r_{11}+r_{12}+r_{13} = \lambda n_3$. Counting the number of vertices adjacent to vertex v in V_2, $r_{21}+2r_{22}+r_{23} = \lambda n_1$ and $r_{21}+2r_{22}+r_{23} = \lambda n_3$. Counting the number of vertices adjacent to vertex v in V_3, $r_{31}+r_{32}+2r_{33} = \lambda n_1$ and $r_{31}+r_{32}+2r_{33} = \lambda n_2$. Therefore, $n_1 = n_2 = n_3$.
Put $n_1 = n_2 = n_3 = n$. Then $b = \lambda n^2/2$, $r = 5\lambda n/6$, $r_{11} = r_{22} = r_{33} = \lambda n/6$ and $r_{12}+r_{13} = r_{21}+r_{23} = r_{31}+r_{32} = 2\lambda n/3$. Thus $\lambda n \equiv 0 \pmod 6$. Since a bowtie is a subgraph of $\lambda K_{n,n,n}$, $n \geq 2$.

(Sufficiency) Case 1. $n \equiv 0 \pmod 6$. Put $n = 6s$. When $s = 1$, let $V_1 = \{1,2,...,6\}$, $V_2 = \{7,8,...,12\}$, $V_3 = \{13,14,...,18\}$.
Construct a balanced bowtie decomposition of $K_{6,6,6}$:
$B_1 = \{(1,7,13),(1,8,14)\}$
$B_2 = \{(2,7,14),(2,8,13)\}$
$B_3 = \{(3,9,15),(3,10,16)\}$
$B_4 = \{(4,9,16),(4,10,15)\}$

$B_5 = \{(5, 11, 17), (5, 12, 18)\}$
$B_6 = \{(6, 11, 18), (6, 12, 17)\}$
$B_7 = \{(7, 15, 5), (7, 16, 6)\}$
$B_8 = \{(8, 15, 6), (8, 16, 5)\}$
$B_9 = \{(9, 17, 1), (9, 18, 2)\}$
$B_{10} = \{(10, 17, 2), (10, 18, 1)\}$
$B_{11} = \{(11, 13, 3), (11, 14, 4)\}$
$B_{12} = \{(12, 13, 4), (12, 14, 3)\}$
$B_{13} = \{(13, 5, 9), (13, 6, 10)\}$
$B_{14} = \{(14, 5, 10), (14, 6, 9)\}$
$B_{15} = \{(15, 1, 11), (15, 2, 12)\}$
$B_{16} = \{(16, 1, 12), (16, 2, 11)\}$
$B_{17} = \{(17, 3, 7), (17, 4, 8)\}$
$B_{18} = \{(18, 3, 8), (18, 4, 7)\}$.
Therefore, $\lambda K_{n,n,n}$ has a balanced bowtie decomposition.

Case 2. $n \equiv 0$ **(mod 3) and** $\lambda \equiv 0$ **(mod 2).** Put $n = 3s$. When $s = 1$, let $V_1 = \{1, 2, 3\}$, $V_2 = \{4, 5, 6\}$, $V_3 = \{7, 8, 9\}$.
Construct a balanced bowtie decomposition of $2K_{3,3,3}$:
$B_1 = \{(1, 4, 7), (1, 5, 8)\}$
$B_2 = \{(2, 5, 8), (2, 6, 9)\}$
$B_3 = \{(3, 6, 9), (3, 4, 7)\}$
$B_4 = \{(4, 8, 3), (4, 9, 1)\}$
$B_5 = \{(5, 9, 1), (5, 7, 2)\}$
$B_6 = \{(6, 7, 2), (6, 8, 3)\}$
$B_7 = \{(7, 3, 5), (7, 1, 6)\}$
$B_8 = \{(8, 1, 6), (8, 2, 4)\}$
$B_9 = \{(9, 2, 4), (9, 3, 5)\}$.
Therefore, $\lambda K_{n,n,n}$ has a balanced bowtie decomposition.

Case 3. $n \equiv 0$ **(mod 2) and** $\lambda \equiv 0$ **(mod 3).** Put $n = 2s$. When $s = 1$, let $V_1 = \{1, 2\}$, $V_2 = \{3, 4\}$, $V_3 = \{5, 6\}$.
Construct a balanced bowtie decomposition of $3K_{2,2,2}$:
$B_1 = \{(1, 3, 5), (1, 4, 6)\}$
$B_2 = \{(2, 3, 6), (2, 4, 5)\}$
$B_3 = \{(3, 5, 1), (3, 6, 2)\}$
$B_4 = \{(4, 5, 2), (4, 6, 1)\}$
$B_5 = \{(5, 1, 3), (5, 2, 4)\}$
$B_6 = \{(6, 1, 4), (6, 2, 3)\}$.
Therefore, $\lambda K_{n,n,n}$ has a balanced bowtie decomposition.

Case 4. $n \geq 2$ **and** $\lambda \equiv 0$ **(mod 6).** Let $V_1 = \{1, 2, ..., n\}$, $V_2 = \{1', 2', ..., n'\}$, $V_3 = \{1'', 2'', ..., n''\}$.
Construct a balanced bowtie decomposition of $6K_{n,n,n}$:
$B_{ij}^{(1)} = \{(i, j', (i + j - 1)''), (i, (j + 1)', (i + j)'')\}$
$B_{ij}^{(2)} = \{(i', j'', i + j - 1), (i', (j + 1)'', i + j)\}$
$B_{ij}^{(3)} = \{(i'', j, (i + j - 1)'), (i'', j + 1, (i + j)')\}$.
Therefore, $\lambda K_{n,n,n}$ has a balanced bowtie decomposition.

References

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[2] W. D. Wallis, Combinatorial Designs. Marcel Dekker, New York and Basel (1988).