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Balanced C_{20}-Trefoil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{20} be *the cycle* on 20 vertices. *The C_{20}-trefoil* is a graph of 3 edge-disjoint C_{20}'s with a common vertex and the common vertex is called *the center of the C_{20}-trefoil.* When K_n is decomposed into edge-disjoint sum of C_{20}-trefoils, it is called that *K_n has a C_{20}-trefoil decomposition.* Moreover, when every vertex of K_n appears in the same number of C_{20}-trefoils, it is called that *K_n has a balanced C_{20}-trefoil decomposition* and this number is called *the replication number.*

2. Balanced C_{20}-trefoil decomposition of K_n

Notation. We denote a C_{20}-trefoil passing through $v_1-v_2-v_3-v_4-v_5-v_6-v_7-v_8-v_9-v_{10}-v_{11}-v_{12}-v_{13}-v_{14}-v_{15}-v_{16}-v_{17}-v_{18}-v_{19}-v_{20}-v_1$, $v_1-v_{21}-v_{22}-v_{23}-v_{24}-v_{25}-v_{26}-v_{27}-v_{28}-v_{29}-v_{30}-v_{31}-v_{32}-v_{33}-v_{34}-v_{35}-v_{36}-v_{37}-v_{38}-v_{39}-v_1$, $v_1-v_{40}-v_{41}-v_{42}-v_{43}-v_{44}-v_{45}-v_{46}-v_{47}-v_{48}-v_{49}-v_{50}-v_{51}-v_{52}-v_{53}-v_{54}-v_{55}-v_{56}-v_{57}-v_{58}-v_1$ by $\{(v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10},v_{11},v_{12},v_{13},v_{14},v_{15},v_{16},v_{17},v_{18},v_{19},v_{20}),(v_1,v_{21},v_{22},v_{23},v_{24},v_{25},v_{26},v_{27},v_{28},v_{29},v_{30},v_{31},v_{32},v_{33},v_{34},v_{35},v_{36},v_{37},v_{38},v_{39}),(v_1,v_{40},v_{41},v_{42},v_{43},v_{44},v_{45},v_{46},v_{47},v_{48},v_{49},v_{50},v_{51},v_{52},v_{53},v_{54},v_{55},v_{56},v_{57},v_{58})\}$.

Theorem 1. K_n has a balanced C_{20}-trefoil decomposition if and only if $n\equiv 1 \pmod{120}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{20}-trefoil decomposition. Let b be the number of C_{20}-trefoils and r be the replication number. Then $b=n(n-1)/120$ and $r=58(n-1)/120$. Among r C_{20}-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{20}-trefoils in which v is the center and v is not the center, respectively. Then $r_1+r_2=r$. Counting the number of vertices adjacent to v, $6r_1+2r_2=n-1$. From these relations, $r_1=(n-1)/120$ and $r_2=19(n-1)/40$. Therefore, $n\equiv 1 \pmod{120}$ is necessary.

(Sufficiency) Put $n=120t+1$. Construct tn C_{20}-trefoils as follows:

$B_i^{(1)}=\{(i,i+1,i+6t+2,i+36t+2,i+54t+3,i+72t+3,i+102t+4,i+24t+3,i+90t+4,i+18t+3,i+81t+4,i+21t+3,i+96t+4,i+27t+3,i+108t+4,i+75t+3,i+60t+3,i+39t+2,i+12t+2,i+3t+1),(i,i+2,i+6t+4,i+36t+3,i+54t+5,i+72t+4,i+102t+6,i+24t+4,i+90t+6,i+18t+4,i+81t+6,i+21t+4,i+96t+6,i+27t+4,i+108t+6,i+75t+4,i+60t+5,i+39t+3,i+12t+4,i+3t+2),(i,i+3,i+6t+6,i+36t+4,i+54t+7,i+72t+5,i+102t+8,i+24t+5,i+90t+8,i+18t+5,i+81t+8,i+21t+5,i+96t+8,i+27t+5,i+108t+8,i+75t+5,i+60t+7,i+39t+4,i+12t+6,i+3t+3)\}$

$B_i^{(2)}=\{(i,i+4,i+6t+8,i+36t+5,i+54t+9,i+72t+6,i+102t+10,i+24t+6,i+90t+10,i+18t+6,i+81t+10,i+21t+6,i+96t+10,i+27t+6,i+108t+10,i+75t+6,i+60t+9,i+39t+5,i+12t+8,i+3t+4),(i,i+5,i+6t+10,i+36t+6,i+54t+11,i+72t+7,i+102t+12,i+24t+7,i+90t+12,i+18t+7,i+81t+12,i+21t+7,i+96t+12,i+27t+7,i+108t+12,i+75t+7,i+60t+11,i+39t+6,i+12t+10,i+3t+5),(i,i+6,i+6t+12,i+36t+7,i+54t+13,i+72t+8,i+102t+14,i+24t+8,i+90t+14,i+18t+8,i+81t+14,i+21t+8,i+96t+14,i+27t+8,i+108t+14,i+75t+8,i+60t+13,i+39t+7,i+12t+12,i+3t+6)\}$

...

$B_i^{(t)}=\{(i,i+3t-2,i+12t-4,i+39t-1,i+60t-3,i+75t,i+108t-2,i+27t,i+96t-2,i+21t,i+87t-2,i+24t,i+102t-2,i+30t,i+114t-2,i+78t,i+66t-3,i+42t-1,i+18t-4,i+6t-2),(i,i+3t-1,i+12t-2,i+39t,i+60t-1,i+75t+1,i+108t,i+27t+1,i+96t,i+21t+1,i+87t,i+24t+1,i+102t,i+30t+1,i+114t,i+78t+1,i+66t-1,i+42t,i+18t-2,i+6t-1),(i,i+3t,i+12t,i+39t+1,i+60t+1,i+75t+$

$2, i+108t+2, i+27t+2, i+96t+2, i+21t+2, i+87t+2, i+24t+2, i+102t+2, i+30t+2, i+114t+2, i+78t+2, i+66t+1, i+42t+1, i+18t, i+6t)\}$ $(i = 1, 2, ..., n)$.
Then they comprise a balanced C_{20}-trefoil decomposition of K_n.

Example 1. Balanced C_{20}-trefoil decomposition of K_{121}.
$B_i = \{(i, i+1, i+8, i+38, i+57, i+75, i+106, i+27, i+94, i+21, i+85, i+24, i+100, i+30, i+112, i+78, i+63, i+41, i+14, i+4),$
$(i, i+2, i+10, i+39, i+59, i+76, i+108, i+28, i+96, i+22, i+87, i+25, i+102, i+31, i+114, i+79, i+65, i+42, i+16, i+5),$
$(i, i+3, i+12, i+40, i+61, i+77, i+110, i+29, i+98, i+23, i+89, i+26, i+104, i+32, i+116, i+80, i+67, i+43, i+18, i+6)\}$ $(i = 1, 2, ..., 121)$.

Example 2. Balanced C_{20}-trefoil decomposition of K_{241}.
$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+208, i+51, i+184, i+39, i+166, i+45, i+196, i+57, i+220, i+153, i+123, i+80, i+26, i+7),$
$(i, i+2, i+16, i+75, i+113, i+148, i+210, i+52, i+186, i+40, i+168, i+46, i+198, i+58, i+222, i+154, i+125, i+81, i+28, i+8),$
$(i, i+3, i+18, i+76, i+115, i+149, i+212, i+53, i+188, i+41, i+170, i+47, i+200, i+59, i+224, i+155, i+127, i+82, i+30, i+9)\}$
$B_i^{(2)} = \{(i, i+4, i+20, i+77, i+117, i+150, i+214, i+54, i+190, i+42, i+172, i+48, i+202, i+60, i+226, i+156, i+129, i+83, i+32, i+10),$
$(i, i+5, i+22, i+78, i+119, i+151, i+216, i+55, i+192, i+43, i+174, i+49, i+204, i+61, i+228, i+157, i+131, i+84, i+34, i+11),$
$(i, i+6, i+24, i+79, i+121, i+152, i+218, i+56, i+194, i+44, i+176, i+50, i+206, i+62, i+230, i+158, i+133, i+85, i+36, i+12)\}$
$(i = 1, 2, ..., 241)$.

Example 3. Balanced C_{2016}-trefoil decomposition of K_{361}.
$B_i^{(1)} = \{(i, i+1, i+20, i+110, i+165, i+219, i+310, i+75, i+274, i+57, i+247, i+66, i+292, i+84, i+328, i+228, i+183, i+119, i+38, i+10),$
$(i, i+2, i+22, i+111, i+167, i+220, i+312, i+76, i+276, i+58, i+249, i+67, i+294, i+85, i+330, i+229, i+185, i+120, i+40, i+11),$
$(i, i+3, i+24, i+112, i+169, i+221, i+314, i+77, i+278, i+59, i+251, i+68, i+296, i+86, i+332, i+230, i+187, i+121, i+42, i+12)\}$
$B_i^{(2)} = \{(i, i+4, i+26, i+113, i+171, i+222, i+316, i+78, i+280, i+60, i+253, i+69, i+298, i+87, i+334, i+231, i+189, i+122, i+44, i+13),$
$(i, i+5, i+28, i+114, i+173, i+223, i+318, i+79, i+282, i+61, i+255, i+70, i+300, i+88, i+336, i+232, i+191, i+123, i+46, i+14),$
$(i, i+6, i+30, i+115, i+175, i+224, i+320, i+80, i+284, i+62, i+257, i+71, i+302, i+89, i+338, i+233, i+193, i+124, i+48, i+15)\}$
$B_i^{(3)} = \{(i, i+7, i+32, i+116, i+177, i+225, i+322, i+81, i+286, i+63, i+259, i+72, i+304, i+90, i+340, i+234, i+195, i+125, i+50, i+16),$
$(i, i+8, i+34, i+117, i+179, i+226, i+324, i+82, i+288, i+64, i+261, i+73, i+306, i+91, i+342, i+235, i+197, i+126, i+52, i+17),$
$(i, i+9, i+36, i+118, i+181, i+227, i+326, i+83, i+290, i+65, i+263, i+74, i+308, i+92, i+344, i+236, i+199, i+127, i+54, i+18)\}$
$(i = 1, 2, ..., 361)$.

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