A-002

Balanced C_{16}-Trefoil Decomposition Algorithm of Complete Graphs

Hideaki Fujimoto Kazuhiko Ushio
Department of Electric and Electronic Engineering Department of Informatics
Faculty of Science and Technology
Kinki University
fujimoto@ele.kindai.ac.jp ushio@info.kindai.ac.jp

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{16} be *the cycle* on 16 vertices. *The C_{16}-trefoil* is a graph of 3 edge-disjoint C_{16}'s with a common vertex and the common vertex is called *the center of the C_{16}-trefoil.* When K_n is decomposed into edge-disjoint sum of C_{16}-trefoils, it is called that *K_n has a C_{16}-trefoil decomposition.* Moreover, when every vertex of K_n appears in the same number of C_{16}-trefoils, it is called that *K_n has a balanced C_{16}-trefoil decomposition* and this number is called *the replication number.*

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_{16}-trefoil decomposition of K_n is $n \equiv 1 \pmod{96}$.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[6, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a C_3-bowtie system.* In this sense, our balanced C_{16}-trefoil decomposition of K_n is to be known as *a balanced C_{16}-trefoil system.*

2. Balanced C_{16}-trefoil decomposition of K_n

Notation. We denote a C_{16}-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_{15} - v_{16} - v_1$, $v_1 - v_{17} - v_{18} - v_{19} - v_{20} - v_{21} - v_{22} - v_{23} - v_{24} - v_{25} - v_{26} - v_{27} - v_{28} - v_{29} - v_{30} - v_{31} - v_1$, $v_1 - v_{32} - v_{33} - v_{34} - v_{35} - v_{36} - v_{37} - v_{38} - v_{39} - v_{40} - v_{41} - v_{42} - v_{43} - v_{44} - v_{45} - v_{46} - v_1$ by $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}), (v_1, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}), (v_1, v_{32}, v_{33}, v_{34}, v_{35}, v_{36}, v_{37}, v_{38}, v_{39}, v_{40}, v_{41}, v_{42}, v_{43}, v_{44}, v_{45}, v_{46})\}$.

Theorem 1. K_n has a balanced C_{16}-trefoil decomposition if and only if $n \equiv 1 \pmod{96}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{16}-trefoil decomposition. Let b be the number of C_{16}-trefoils and r be the replication number. Then $b = n(n-1)/96$ and $r = 46(n-1)/96$. Among r C_{16}-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{16}-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/96$ and $r_2 = 45(n-1)/96$. Therefore, $n \equiv 1 \pmod{96}$ is necessary.

(Sufficiency) Put $n = 96t + 1$. Construct tn C_{16}-trefoils as follows:
$B_i^{(1)} = \{ (i, i+1, i+6t+2, i+36t+2, i+54t+3, i+72t+3, i+6t+3, i+66t+3, i+21t+3, i+69t+3, i+12t+3, i+75t+3, i+60t+3, i+39t+2, i+12t+2, i+3t+1), (i, i+2, i+6t+4, i+36t+3, i+54t+5, i+72t+4, i+6t+5, i+66t+4, i+21t+5, i+69t+4, i+12t+5, i+75t+4, i+60t+5, i+39t+3, i+12t+4, i+3t+2), (i, i+3, i+6t+6, i+36t+4, i+54t+7, i+72t+5, i+6t+7, i+66t+5, i+21t+7, i+69t+5, i+12t+7, i+75t+5, i+60t+7, i+39t+4, i+12t+6, i+3t+3) \}$
$B_i^{(2)} = \{ (i, i+4, i+6t+8, i+36t+5, i+54t+9, i+72t+6, i+6t+9, i+66t+6, i+21t+9, i+69t+6, i+12t+9, i+75t+6, i+60t+9, i+39t+5, i+12t+8, i+3t+4), (i, i+5, i+6t+10, i+36t+6, i+54t+11, i+72t+7, i+6t+11, i+66t+7, i+21t+11, i+69t+7, i+12t+11, i+75t+7, i+60t+11, i+39t+6, i+12t+10, i+3t+5), (i, i+6, i+6t+12, i+36t+7, i+54t+13, i+72t+8, i+6t+13, i+66t+8, i+21t+13, i+69t+8, i+12t+13, i+75t+8, i+60t+13, i+39t+7, i+12t+12, i+3t+6) \}$

...
$B_i^{(t)} = \{ (i, i+3t-2, i+12t-4, i+39t-1, i+60t-3, i+75t, i+12t-3, i+69t, i+27t-3, i+72t, i+18t-3, i+78t, i+66t-3, i+42t-1, i+18t-4, i+6t-2),$
$(i, i+3t-1, i+12t-2, i+39t, i+60t-1, i+75t+1, i+12t-1, i+69t+1, i+27t-1, i+72t+1, i+18t-1, i+78t+1, i+66t-1, i+42t, i+18t-2, i+6t-1),$
$(i, i+3t, i+12t, i+39t+1, i+60t+1, i+75t+2, i+12t+1, i+69t+2, i+27t+1, i+72t+2, i+18t+1, i+78t+2, i+66t+1, i+42t+1, i+18t, i+6t) \}$
$(i = 1, 2, ..., n)$.

Then they comprise a balanced C_{16}-trefoil decomposition of K_n.

Example 1. Balanced C_{16}-trefoil decomposition of K_{97}.

$B_i = \{(i, i+1, i+8, i+38, i+57, i+75, i+9, i+69, i+24, i+72, i+15, i+78, i+63, i+41, i+14, i+4),$
$(i, i+2, i+10, i+39, i+59, i+76, i+11, i+70, i+26, i+73, i+17, i+79, i+65, i+42, i+16, i+5),$
$(i, i+3, i+12, i+40, i+61, i+77, i+13, i+71, i+28, i+74, i+19, i+80, i+67, i+43, i+18, i+6)\}$
$(i = 1, 2, ..., 97)$.

Example 2. Balanced C_{16}-trefoil decomposition of K_{193}.

$B_i^{(1)} = \{(i, i+1, i+14, i+74, i+111, i+147, i+15, i+135, i+45, i+141, i+27, i+153, i+123, i+80, i+26, i+7),$
$(i, i+2, i+16, i+75, i+113, i+148, i+17, i+136, i+47, i+142, i+29, i+154, i+125, i+81, i+28, i+8),$
$(i, i+3, i+18, i+76, i+115, i+149, i+19, i+137, i+49, i+143, i+31, i+155, i+127, i+82, i+30, i+9)\}$
$B_i^{(2)} = \{(i, i+4, i+20, i+77, i+117, i+150, i+21, i+138, i+51, i+144, i+33, i+156, i+129, i+83, i+32, i+10),$
$(i, i+5, i+22, i+78, i+119, i+151, i+23, i+139, i+53, i+145, i+35, i+157, i+131, i+84, i+34, i+11),$
$(i, i+6, i+24, i+79, i+121, i+152, i+25, i+140, i+55, i+146, i+37, i+158, i+133, i+85, i+36, i+12)\}$
$(i = 1, 2, ..., 193)$.

Example 3. Balanced C_{16}-trefoil decomposition of K_{289}.

$B_i^{(1)} = \{(i, i+1, i+20, i+110, i+165, i+219, i+21, i+201, i+66, i+210, i+39, i+228, i+183, i+119, i+38, i+10),$
$(i, i+2, i+22, i+111, i+167, i+220, i+23, i+202, i+68, i+211, i+41, i+229, i+185, i+120, i+40, i+11),$
$(i, i+3, i+24, i+112, i+169, i+221, i+25, i+203, i+70, i+212, i+43, i+230, i+187, i+121, i+42, i+12)\}$
$B_i^{(2)} = \{(i, i+4, i+26, i+113, i+171, i+222, i+27, i+204, i+72, i+213, i+45, i+231, i+189, i+122, i+44, i+13),$
$(i, i+5, i+28, i+114, i+173, i+223, i+29, i+205, i+74, i+214, i+47, i+232, i+191, i+123, i+46, i+14),$
$(i, i+6, i+30, i+115, i+175, i+224, i+31, i+206, i+76, i+215, i+49, i+233, i+193, i+124, i+48, i+15)\}$
$B_i^{(3)} = \{(i, i+7, i+32, i+116, i+177, i+225, i+33, i+207, i+78, i+216, i+51, i+234, i+195, i+125, i+50, i+16),$
$(i, i+8, i+34, i+117, i+179, i+226, i+35, i+208, i+80, i+217, i+53, i+235, i+197, i+126, i+52, i+17),$
$(i, i+9, i+36, i+118, i+181, i+227, i+37, i+209, i+82, i+218, i+55, i+236, i+199, i+127, i+54, i+18)\}$
$(i = 1, 2, ..., 289)$.

References

[1] C. J. Colbourn and A. Rosa, Triple Systems. Clarendom Press, Oxford, 1999.
[2] P. Horák and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria*, Vol. 26, pp. 91–105, 1988.
[3] K. Ushio and H. Fujimoto, Balanced bowtie and trefoil decomposition of complete tripartite multigraphs, *IEICE Trans. Fundamentals*, Vol. E84-A, No. 3, pp. 839–844, March 2001.
[4] K. Ushio and H. Fujimoto, Balanced foil decomposition of complete graphs, *IEICE Trans. Fundamentals*, Vol. E84-A, No. 12, pp. 3132–3137, December 2001.
[5] K. Ushio and H. Fujimoto, Balanced bowtie decomposition of complete multigraphs, *IEICE Trans. Fundamentals*, Vol. E86-A, No. 9, pp. 2360–2365, September 2003.
[6] W. D. Wallis, Combinatorial Designs. Marcel Dekker, New York and Basel, 1988.