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A semigroup of homomorphisms based on vertex connectivity of weighted directed graphs

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1. Introduction

In this paper we give our definition of homomorphisms of general weighted directed graphs and investigate the semigroups of surjective homomorphims and synthesize graphs to obtain a generator of pricipal left (or right) ideal in the semigroup. This study is motivated by reducing the redundancy in concurrent systems, for example, Petri nets which are represented by weighted bipartite graphs. Here we can more simply obtain some results in weighted direct graphs that is similar to the ones shown in Petri nets[11]

In a general weighted directed graph, weights given to edges are mesured by some quantity, for example, usually nonnegative integers. Here slightly extending the notion of weight, we adopt a kind of ring R as this quantity. For weighted digraphs $(V_i, E_i, W_i)(i = 1, 2)$, a usual graph homomorphism $\phi : V_1 \to V_2$ satisfies $W_2(\phi(u), \phi(v)) = W_1(u, v)$ to preserve adjacencies of the graphs. Whereas we originally extend this definition slightly and our homomorphism is defined by the pair (ϕ, ρ) based on the similarity of the edge connection. (ϕ, ρ) satisfies $W_2(\phi(u), \phi(v)) = \rho(u)\rho(v)W_1(u, v)$, where $\rho : V_1 \to R$ and R is a principal ideal domain. When this equality holds among two weighted digraphs, interestingly the stuctures of these two graphs can be explained in terms of the similarity equivalence.

2. Preliminaries

Here we introduce an extension of homomorphisms of a usual weighted directed graph and state some properties of the semigroup of these homomorphisms.

2.1. Graphs and Homomorphisms

In a general weighted directed graph, weights given to edges are mesured by some quantity, for example, usually nonnegative integers. Here slightly extending the notion of weight, we adopt a kind of ring R as this quantity. More precisely we assume that $(R, +, \cdot)$ has at least two distinct elements $0, 1 \in R$ and satisfies a ring condition (i) to (iii):

(i) $(R, +, 0)$ is an abelean group.
(ii) $(R, \cdot, 1)$ is a monoid.
(iii) $(R, +, \cdot)$ satisfies the distributive laws.

Moreover through the manuscript we assume that R is a principal ideal domain (abbreviated as PID)[9], that is, satisfies the following (iv), (v) and (vi).

(iv) $(R, \cdot, 1)$ is a commutative monoid.
(v) $ab = 0$ implies $a = 0$ or $b = 0$.
(vi) Every ideal I in R is principal, that is, $I = RaR$ for some $a \in R$.

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We require the conditions (iv) and (v) that R is a domain, for defining the quotient field $Q(R) = \{r/s | r, s \in R, s \neq 0\}$ of R by $Q(R)$, which is the smallest field containing a domain R.

By (vi), for any nonempty $S = \{a_1, a_2, \ldots, a_n\} \subset R$, there exists $a \in R$ such that $a_1R \cup a_2R \cup \cdots \cup a_nR = aR$, which is called a greatest common divisor of S. The set of all the greatest common divisors of S is denoted by $\gcd(S)$.

Definition 1. A *weighted directed graph* (weighted digraph, for short) is a 3-tuple (V, E, W) where
(1) V is a finite set of vertices,
(2) $E\,(\subset V \times V)$ is a set of edges,
(3) $W : E \to R$ is a *weight function*, where R is a PID. □

According to custom, $(u, v) \in E \iff W(u, v) \neq 0$.

Definition 2. Let $G_1 = (V_1, E_1, W_1)$ and $G_2 = (V_2, E_2, W_2)$ be weighted digraphs. Then a pair (ϕ, ρ) is called a *(weak weight preserving) homomorphism* (for short, *w-homomorphism*) from G_1 to G_2 if $W_i : E_i \to R$ have the same image R and the maps $\phi : V_1 \to V_2$, $\rho : V \to Q(R)$ satisfy the condition that for any $u, v \in V_1$,

$$W_2(\phi(u), \phi(v)) = \rho(u)\rho(v)W_1(u, v), \qquad (1)$$

where $Q(R)$ is the quotient field of R. Especially if $\rho = 1_V$, i.e., $\rho(u) = 1$ for any $u \in V$, then w-homomorphism is called a *strictly weight preserving homomorphism* (s-homomorphism, for short). □

Example 1. Let $G_i = (V_i, E_i, W_i)\,(i = 1, 2)$ be the weighted digraphs depicted in Fig.1, $W_i : V_i \to \boldsymbol{Z}$ the weight functions, where $\boldsymbol{Z}$ is the set of integers but we don't use its negative part. That is,

$V_1 = \{u_1, u_2, v_1, v_2\}, V_2 = \{u_3, u_4, v_3\}$,
$E_1 = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2)\}$
$E_2 = \{(u_3, v_3), (u_4, v_3)\}$
$W_1 : (u_1, v_1) \mapsto 1, (u_1, v_2) \mapsto 2, (u_2, v_1) \mapsto 3, (u_2, v_2) \mapsto 6$
$W_2 : (u_3, v_3) \mapsto 3, (u_4, v_3) \mapsto 9.$

(a) Weighted Digraph G_1 (b) Weighted Digraph G_2

Fig. 1: Weighted Digraph G_1 and G_2 with $G_1 \sqsupseteq G_2$.

The following (ϕ_1, ρ_1) is a w-homomorphism from G_1 to G_2.

$$\phi_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u_3 & u_4 & v_3 & v_3 \end{pmatrix}, \ \rho_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 1 & 1 & 3 & 3/2 \end{pmatrix}.$$

□

A w-homomorphism (ϕ, ρ) is called *injective* (resp. *surjective*) if ϕ is injective (resp. surjective). In particular, it is called a *w-isomorphism* from G_1 to G_2 if it is injective and surjective. Then G_1 is said to be *w-isomorphic* to G_2 and we write $G_1 \simeq_{\mathrm{w}} G_2$. Moreover, in case of $G_1 = G_2 = G$, a w-isomorphism is called a *w-automorphism* of G. By $\mathbf{Aut}_{\mathrm{w}}(G)$ we denote the set of all the w-automorphisms of G. Similarly s-isomorphism $\simeq_{\mathrm{s}}$ s-automorphism and $\mathbf{Aut}_{\mathrm{s}}(G)$ are defined.

We obtain the following theorem, which is similar to the result in [10].

Theorem 1. *For a given finite group H, there exists a graph G such that $\mathbf{Aut}_s(G) \simeq H$.* □

Proof) Let $H = \{g_1 = e, g_2, \ldots, g_n\}$ with the identity $g_1 = e$ and $p_1, p_2, \ldots, p_n$ be distinct integers. We define the graph $G = (H, E, W)$ by for any $g, h \in H$ $W(g, h) = p_i$ if $h = g_i g$ holds.

Then $(\phi_k, \mathbf{1})$ is an s-automorphism of G, where $\phi_k : G \to G, g \mapsto g g_k \, (1 \leq k \leq n)$ and $\mathbf{1} : G \to \boldsymbol{Z}, g \mapsto 1$ for any $g \in G$. Indeed, let any $g, h \in H$ with $h = g_i g$.

$$W(\phi_k(g), \phi_k(h)) = W(g g_k, g_i g g_k) = p_i = W(g, h)$$

Conversely, suppose $(\phi_k, \mathbf{1})$ is an s-automorphism of G and let $g_k = \phi(e)$. Then since $(\phi_k, \mathbf{1})$ strictly preserves the weights of edges in G, for any $g_i \in H$,

$$W(e, g_i) = p_i = W(\phi(e), \phi(g_i)) = W(g_k, \phi(g_i))$$

holds. By the construction of the weight function, we have $\phi(g_i) = g_i g_k$ and thus $\phi = \phi_k$

Definitely $(\phi_k, \mathbf{1})$ corresponds to $g_k \in H$ because $\phi_j \phi_k \ : \ g \ \mapsto \ g(g_j g_k)$ holds. Thus $\mathbf{Aut}_s(G) = \{\phi_1 \ldots, \phi_n\} \simeq H$ □

2.2. Composition of the w-homomorphisms

We define the composition of the w-homomorphisms. In this manuscript, we write $\phi\psi$ for the composition $\psi \circ \phi$ of maps.

Definition 3. Let $G_i = (V_i, E_i, W_i) \, (i = 1, 2, 3)$ be weighted digraphs, $(\phi, \rho) : G_1 \to G_2$ and $(\psi, \sigma) : G_2 \to G_3$ be w-homomorphisms. Then the composition of these w-homomorphisms are defined by the semidirect product

$$(\phi, \rho)(\psi, \sigma) \overset{\mathrm{def}}{=} (\phi, \rho) \rtimes (\psi, \sigma) = (\phi\psi, \rho \otimes (\phi\sigma)),$$

where $\rho \otimes (\phi\sigma) : V \to Q(R), u \mapsto \rho(u)\sigma(\phi(u))$. □

Indeed, checking the validity of the definition.

$$\begin{aligned} & W_3(\psi(\phi(u)), \psi(\phi(v))) \\ & = \sigma(\phi(u))\sigma(\phi(v))W_2(\phi(u), \phi(v)) \\ & = \sigma(\phi(u))\sigma(\phi(v))\rho(u)\rho(v)W_1(u, v) \\ & = \sigma(\phi(u))\rho(u)\sigma(\phi(v))\rho(v)W_1(u, v) \\ & = (\rho \otimes (\phi\sigma))(u)(\rho \otimes (\phi\sigma))(v)W_1(u, v) \end{aligned}$$

hold.

Example 2. Let $G_i = (V_i, E_i, W_i) \, (i = 2, 3)$ be weighted digraphs depicted in Fig.2. The following (ϕ_1, ρ_1) is the w-homomorphism from G_1 to G_2 in Example 1. (ϕ_2, ρ_2) is a w-homomorphism from G_2 to G_3.

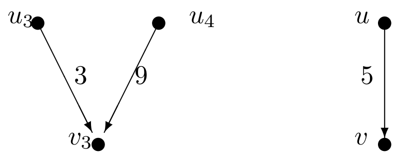
$$\phi_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u_3 & u_4 & v_3 & v_3 \end{pmatrix}, \rho_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 1 & 1 & 3 & 3/2 \end{pmatrix},$$
$$\phi_2 = \begin{pmatrix} u_3 & u_4 & v_3 \\ u & u & v \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} u_3 & u_4 & v_3 \\ 5/3 & 5/9 & 1 \end{pmatrix}.$$

We have

$$\phi_1\rho_2 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 5/3 & 5/9 & 1 & 1 \end{pmatrix}.$$

Therefore, $(\phi, \rho) = (\phi_1\phi_2, \rho_1 \otimes (\phi_1\rho_2)) = (\phi_1, \rho)(\phi_2, \rho_2)$ is the composition of them, where

$$\phi = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u & u & v & v \end{pmatrix}, \ \rho = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 5/3 & 5/9 & 3 & 3/2 \end{pmatrix}.$$



(b) Weighted Digraph G_2 (c) Weighted Digraph G_3

Fig. 2: Weighted Digraphs G_2 and G_3.

□

Immediately, we obtain the following lemma regarding to $\oplus$.

Lemma 1. Let ϕ and ψ be arbitrary maps on V and $f, g : V \to Q(R)$. $\mathbf{1}_V$ means the constant mapping defined by $\mathbf{1}_V : V \to Q(R), v \mapsto 1$, f^{-1} means the mapping $V \to Q(R), v \mapsto 1/f(v)$. Then the following equations are true.

(1) $(\phi\psi)f = \phi(\psi f)$.
(2) $\phi(f \otimes g) = (\phi f) \otimes (\phi g)$.
(3) $\psi e_V = e_V$.
(4) $(\phi f) \otimes (\phi f^{-1}) = e_V$.
(5) $(\phi f)^{-1} = \phi f^{-1}$.

Proof) We can easily verify the equations. □

For weighted digraphs G_1 and G_2, we write $G_1 \sqsupseteq G_2$ if there exists a surjective w-homomorphism from G_1 to G_2. The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order $\sqsupseteq$ is regarded as an order up to w-isomorphism.

Proposition 1. Let G_1, G_2, G_3 be weighted digraphs. Then,
(1) $G_1 \sqsupseteq G_1$.
(2) $G_1 \sqsupseteq G_2$ and $G_2 \sqsupseteq G_1 \iff G_1 \simeq G_2$.
(3) $G_1 \sqsupseteq G_2$ and $G_2 \sqsupseteq G_3$ imply $G_1 \sqsupseteq G_3$.

Proof) We can easily verify the inequalities. □

Remark that in Example 2, ϕ_1 and ϕ_2 are sujective, $\phi_1\phi_2$ is also. Therefore $G_1 \sqsupseteq G_2 \sqsupseteq G_3$ holds.

2.3.Similarity of vertices

Similarity of vertices means that vertices can be reduced to one vertex by some w-homomorphism.

Definition 4 (Similar)**.** Let $G = (V, E, W)$ be a weighted digraph. Two vertices $u, v \in V$ are said to be *similar* if there exist some $s, t \in R \setminus \{0\}$ such that $sW(u, x) = tW(v, x)$ and $sW(x, u) = tW(x, v)$ for all $x \in V$. □

The similarity defined above forms obviously an equivalence relation on V. We denote this relation by $\sim_G$ (or simply $\sim$) and the $\sim_G$-class of a vertex u by $C(u)$. A vertex u is said to be *isolated* if u has no connection, that is, $W(u, x) = W(x, u) = 0$ to any $x \in V$.

Note that any two isolated vertices u and v are similar because for any element $s \neq 0$ in R, $sW(u, x) = 0 = sW(v, x)$ and $sW(x, u) = 0 = sW(x, v)$for all $x \in V$.

Proposition 2. *Let $G = (V, E, W)$ be a weighted digraph. The following conditions are equivalent.*
(1) u and v are similar.
(2) There exist a graph $G' = (V', E', W')$ and a surjective w-homomorphism (ϕ, ρ) from G to G' such that $\phi(u) = \phi(v)$.

Proof) (1)⇒ (2) We will construct the graph $G' = (V', E', W')$ as follows: $V' = V \setminus \{v\}$, $E' = E \cap (V' \times V')$, and $W' = W|(V' \times V')$(the restriction of W to $V' \times V'$). Since u and v are similar, there exist $s, t \in R \setminus \{0\}$ such that $sW(u, x) = tW(v, x)$ and $sW(x, u) = tW(x, v)$ for all $x \in V$. Then we define the w-homomorphism $(\phi, \rho) : G \to G'$ as follows:

$$\begin{aligned}
&\phi(u) = u, \rho(u) = 1,\\
&\phi(v) = u, \rho(v) = t/s,\\
&\phi(x) = x, \rho(x) = 1 \in R \text{ if } x \in V \setminus \{u, v\}.
\end{aligned}$$

We can verify for the w-homomorphism (ϕ, ρ) to preserve the weight functions.

(2)⇒(1) Let $\rho : P \to Q(R)$. By the definition of w-homomorphism , $W'(\phi(u), \phi(x)) = \rho(u)\rho(x)W(u, x) = \rho(v)\rho(x)W(v, x) = W'(\phi(v), \phi(x))$ for any $x \in V$. We have

$$\begin{aligned}
&\rho(u)W(u, x) = \rho(v)W(v, x), \text{ and similarly}\\
&\rho(u)W(x, u) = \rho(v)W(x, v).
\end{aligned}$$

Then since we can write $\rho(u) = s'/s'' \in Q(R)$ and $\rho(v) = t'/t'' \in Q(R)$, setting $s = s't''$ and $t = t's''$ we conclude that u and v are similar. □

3. Ideals in the semigroup $\mathcal{S}$

In this section we define the set $\mathcal{S}$ of all surjective w-homomorphisms between two weighted digraphs and a (extra) zero element 0. Introducing the multiplication by the composition, $\mathcal{S}$ forms a semigroup,

For a surjective w-homomorphim $x : G_1 \to G_2$, G_1 is called the domain of x, denoted by $Dom(x)$, and G_2 is called the image(or range) of x, denoted by $Im(x)$. Especially $Dom(0) = Im(0) = \emptyset$. The multiplication of $x = (\phi, \rho)$ and $y = (\psi, \sigma)$ is defined by

$$x \cdot y \overset{\text{def}}{=} \begin{cases} (\phi\psi, (\phi\rho) \otimes \sigma) & \text{if} \quad Im(x) = Dom(y). \\ 0 & \text{otherwise.} \end{cases}$$

3.1.Green's equivalences on the semigroup $\mathcal{S}$

For convenience of notation, $\mathcal{S}^1 = \mathcal{S} \cup \{1\}$ is the monoid obtained from a semigroup $\mathcal{S}$ by adjoining an (extra) identity 1, that is, $1 \cdot s = s \cdot 1 = s$ for all $s \in \mathcal{S}$ and $1 \cdot 1 = 1$.

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a semigroup S, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$\begin{aligned}
&x\mathcal{L}y \iff S^1x = S^1y,\\
&x\mathcal{R}y \iff xS^1 = yS^1,\\
&x\mathcal{J}y \iff S^1xS^1 = S^1yS^1,\\
&\mathcal{H} = \mathcal{L} \cap \mathcal{R},\\
&\mathcal{D} = (\mathcal{L} \cup \mathcal{R})^*,
\end{aligned}$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. S^1x (resp. xS^1) is called the *principal left* (resp. *right*) *ideal generated by* x and S^1xS^1 the it principal (two-sided) ideal generated by x. Then, the following facts are generally true[7, 3].

Fact 1. *The following relations are true.*

$$\begin{aligned}
&(1)\ \mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}\\
&(2)\ \mathcal{H} \subset \mathcal{L}\ \ (\textit{resp.}\ \mathcal{R})\ \subset \mathcal{D} \subset \mathcal{J}
\end{aligned}$$

Fact 2. *An $\mathcal{H}$-class is a group if and only if it contains an idempotent e, that is $e^2 = e$.*

Now we consider the case of $S = \mathcal{S}$ in the rest of the maniscript. The following lemma is obviously true.

Lemma 2. *Let* $x : G_1 \to G_2$, $y : G_3 \to G_4 \in \mathcal{S}$. *Then,*
(1) $x\mathcal{S}^1 \subset y\mathcal{S}^1 \Longrightarrow G_1 = G_3 \sqsupseteq G_4 \sqsupseteq G_2$.
(2) $\mathcal{S}^1x \subset \mathcal{S}^1y \Longrightarrow G_1 \sqsupseteq G_3 \sqsupseteq G_2 = G_4$.
(3) $x\mathcal{S}^1 = y\mathcal{S}^1 \Longrightarrow G_1 = G_3$ *and* $G_2 \simeq_{\mathrm{w}} G_4$.
(4) $\mathcal{S}^1x = \mathcal{S}^1y \Longrightarrow G_1 \simeq_{\mathrm{w}} G_3$ *and* $G_2 = G_4$. □

Remark that any reverses of the implications above are not necessarily true.

Proposition 3. The following conditions are equivalent.
(1) H is an $\mathcal{H}$-class and a group.
(2) $H = \mathbf{Aut}_{\mathrm{w}}(G)$ for some weighted digraph G.

Proof) (1)⟹(2) By Fact2, H contains an idempotent e, that is $e^2 = e$. This implies $Dom(e) = Im(e) = G$ for some weighted digraph G. By (3) and (4) of Lemma 2, $Dom(x) = Dom(e) = G$ and $Im(x) = Im(e) = G$ for any $x \in H$ because $x\mathcal{S}^1 = e\mathcal{S}^1$ and $\mathcal{S}^1x = \mathcal{S}^1e$ hold. Therefore each element of H is a w-automorphism of G. Conversely, for a w-automorphism x of G, $x \in H$ because x is a surjective morphism with $Dom(x) = Im(x) = G$. Hence we have $H = \mathbf{Aut}_{\mathrm{w}}(G)$.
(2)⟹(1) For $x, y \in H = \mathbf{Aut}_{\mathrm{w}}(G)$, there exist $z, w \in H$ such that $x = zy$ and $wx = y$. This implies $\mathcal{S}^1x = \mathcal{S}^1y$. Similarily we have $x\mathcal{S}^1 = y\mathcal{S}^1$. Therefore $x\mathcal{H}y$. Conversely, $x\mathcal{H}y$ and $x \in H$ imply $y \in H$ because y is a surjective w-homomorphism with $Dom(y) = Im(y) = G$. Hence H is an $\mathcal{H}$-class and a group. □

Proposition 4. On the semigroup $\mathcal{S}$, $\mathcal{J} = \mathcal{D}$.

Proof) Since $\mathcal{D} \subset \mathcal{J}$ holds, it is enough to show the reverse inclusion.

$$\begin{aligned} x\mathcal{J}y &\iff \mathcal{S}^1x\mathcal{S}^1 = \mathcal{S}^1y\mathcal{S}^1 \\ &\iff \exists u, v, z, w \in \mathcal{S}^1\ (x = uyv, y = zxw) \end{aligned}$$

implies that $x = uzxwv, y = zuyvw$. Setting $P = Dom(x)$,$Q = Dom(y)$,$R = Im(x)$ and $S = Im(y)$, $uz : P \to P$, $zu : Q \to Q$, $wv : R \to R$, $vw : S \to S$ are w-automorphisms. This implies that u, v, z, w are w-isomorphisms. Let $t = xw$. Then,

$$\begin{aligned} &x = x(ww^{-1}) = (xw)w^{-1} = tw^{-1} \\ &y = z(xw) = zt \\ &t = (z^{-1}z)t = z^{-1}(zt) = z^{-1}y \end{aligned}$$

Therefore $x\mathcal{S}^1 = t\mathcal{S}^1$ and $\mathcal{S}^1t = \mathcal{S}^1y$, that is, $x\mathcal{R}t\mathcal{L}y$. Thus $\mathcal{D} \subset \mathcal{J}$. □

3.2.Intersection of principal ideals

The aim here is that for given $x, y \in \mathcal{S}$ we find a elements z such that $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$ (resp. $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$). $x\mathcal{S}^1 \cap y\mathcal{S}^1 = \{0\}$ (resp. $\mathcal{S}^1x \cap \mathcal{S}^1y = \{0\}$) is a trivial case($z = 0$). We should only consider the non-trivial case. For a surjective map $\phi : V_1 \to V_2$, we denote the equivalence relation $\phi\phi^{-1} = \{(u, v) | v \in \phi\phi^{-1}(u)\}$ on V_1 by $\ker\phi$, that is, the set of all pairs of vertices which map to the same image by ϕ.

Lemma 3. Let $G_i = (V_i, E_i.W_i)(i = 1, 2, 3)$ be weighted graphs, $x = (\phi, \rho) : G_1 \to G_3$, $y = (\psi, \sigma) : G_2 \to G_3$ be elements of $\mathcal{S}$. If $|\phi^{-1}(u)| \le |\psi^{-1}(u)|$ for any $u \in V_3$, then $\mathcal{S}^1y \subset \mathcal{S}^1x$.

Proof) By the assumption, we can choose some surjective morphism $\xi : V_2 \to V_1$ such that $\xi(\psi^{-1}(u)) = \phi^{-1}(u)$ for any $u \in V_3$.

$$W_1(\xi(u), \xi(v)) = \frac{\sigma(u)\sigma(v)}{\rho(\xi(u))\rho(\xi(v))}W_2(u, v).$$

So $\tau : V_2 \to Q(R)$ is defined by $\tau = \sigma \otimes (\xi\rho)^{-1}$. Then, we can verify that (ξ, τ) is a surjective morphism from G_2 to G_1 and thus $z \in \mathcal{S}^1$, $y = zx$. Therefore $\mathcal{S}^1y \subset \mathcal{S}^1x$. □

Remark that enumerating all the surjective maps such as ξ in the proof, the number N of them is represented as

$$N = \prod_{i=1}^{k}(s_{n_i}^{m_i} \times m_i!),$$

where $V_3 = \{u_1, u_2, \ldots, u_k\}$, $m_i = |\phi^{-1}(u_i)|$, $n_i = |\psi^{-1}(u_i)|$, and $s_{n_i}^{m_i}$ is the Stirling number (of the second kind). $s_{n_i}^{m_i}(n_i \ge m_i)$ is the number of partitions of a set of n_i objects into m_i classes[1].

Lemma 4. Let $G_i = (V_i, E_i.W_i)(i = 0, 1, 2)$ be weighted digraphs, $x = (\phi, \rho) : G_0 \to G_1$, $y = (\psi, \sigma) : G_0 \to G_2$ be elements of $\mathcal{S}$. If $\ker\phi \subset \ker\psi$, then $y\mathcal{S}^1 \subset x\mathcal{S}^1$.

Proof) Let u, v be arbitrary elements of V_1, respectively. By the assumption, $\bar{u}, \bar{v} \in V_2$ are uniquely determined and let

$$\begin{aligned} \phi^{-1}(u) &= \{u_1, u_2, \ldots, u_n\} \subset \psi^{-1}(\bar{u}), \\ \phi^{-1}(v) &= \{v_1, v_2, \ldots, v_m\} \subset \psi^{-1}(\bar{v}), \end{aligned}$$

Then we can easily check that

$$\begin{aligned} &W_1(u, v) = W_1(\phi(u_i), \phi(v_j)) = \rho(u_i)\rho(v_j)W_0(u_i, v_j), \\ &W_2(\bar{(u)}, \bar{(v)}) = W_2(\psi(u_i), \psi(v_j)) = \sigma(u_i)\sigma(v_j)W_0(u_i, v_j), \end{aligned}$$

for any $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The right hand sides of the equations above are constants not depending on i and j. So

$$\begin{aligned} &\xi : V_1 \to V_2,\ u \mapsto \bar{u}, \text{ where } \phi^{-1}(u) \subset \psi^{-1}(\bar{u}), \text{ and} \\ &\nu : V_1 \to Q(R),\ u \mapsto \sigma(u_i)\rho^{-1}(u_i), \text{ where } \phi(u_i) = u \end{aligned}$$

are well-defined. Therefore we have $z = (\xi, \nu) \in \mathcal{S}$ and thus $y = xz$, that is, $y\mathcal{S}^1 \subset x\mathcal{S}^1$. □

Proposition 5 (Intersection of Principal Left Ideals)**.** Let $G_i = (V_i, E_i, W_i)(i = 1, 2, 3)$ be weighted digraphs, $x = (\phi_1, \rho_1)) : G_1 \to G_3$, $y = (\phi_2, \rho_2)) : G_2 \to G_3$ be elements of $\mathcal{S}$, $V_3 = \{u_1, u_2, \ldots, u_N\}$. Let

$$n_i = \max\{|\phi_1^{-1}(u_i)|, |\phi_2^{-1}(u_i)|\} \text{ for each } i = 1, 2, \ldots, N.$$

Taking sets $U_1, U_2, \ldots, U_N$ with their sizes $|U_i| = n_i (i = 1, 2, \ldots, N)$, we construct a weighted digraph $G = (V, E, W)$, where $V = \bigcup_{1 \le i \le N} U_i$ and for any $u, v \in V$,

$$W(u,v) = W_3(u_i, u_j) \quad \text{if } u \in U_i, v \in U_j,$$

Then, $z = (\phi, \mathbf{1}_{\otimes_V}) : G \to G_3$, where $\phi : U_i \ni u \mapsto u_i$ and $\mathbf{1}_{\otimes_V} : V \to Q(R), v \mapsto 1 \in Q(R)$, is a surjective morphism. Moreover, $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \mathcal{S}^1 z$.

Proof) By Lemma 3 and the construction of G, $z = ax = by$ for some $a, b \in \mathcal{S}^1$. Therefore $z \in \mathcal{S}^1 x \cap \mathcal{S}^1 y$.

Conversely we show that $w = (\psi, \sigma) \in \mathcal{S}^1 x \cap \mathcal{S}^1 y$ implies $w \in \mathcal{S}^1 z$. We can write $w = a'x = b'y$ for some $a', b' \in \mathcal{S}^1$. Let $u_i \in V_3$. In our construction, $|\phi^{-1}(u_i)| = \max\{|\phi_1^{-1}(u_i)|, |\phi_2^{-1}(u_i)|\}$. Since $w = a'x = b'y$ holds, we have $|\phi_1^{-1}(u_i)| \le |\psi^{-1}(u_i)|$ and $|\phi_2^{-1}(u_i)| \le |\psi^{-1}(u_i)|$ and thus $|\phi^{-1}(u_i)| \le |\psi^{-1}(u_i)|$. By Lemma 3, we conclude $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \mathcal{S}^1 z$. □

Corollary 1 (Diamond Property I)**.** Let $G_i = (V_i, E_i, W_i)$ $(i = 1, 2, 3)$ be weighted digraphs with $G_1 \sqsupseteq G_3$ and $G_2 \sqsupseteq G_3$. Then there exists a weighted digraph G such that $G \sqsupseteq G_1$ and $G \sqsupseteq G_2$.

We consider the intersection of two principal right ideals. The case of principal right ideals is rather difficult compared to that of principal left ideals.

$(ker\phi \cup \ker\psi)^*$ is the smallest equivalence relation on V which includes both $\ker\phi$ and $\ker\psi$, that is, it is the reflexive and transitive closure of $ker\phi \cup \ker\psi$.

Proposition 6 (Intersection of Principal Right Ideals)**.** Let $G_i = (V_i, E_i.W_i) (i = 0, 1, 2)$ be weighted digraphs, $x = (\phi_1, \rho_1) : G_0 \to G_1$, $y = (\phi_2, \rho_2) : G_0 \to G_2$ be elements of $\mathcal{S}$. Let $C_1, C_2, \ldots, C_N$ be all the $(\ker\phi_1 \cup \ker\phi_2)^*$-classes in V_0.

$\rho : V_0 \to Q(R)$ is defined by if u is 0-isolated then $\rho(u) = 1$ and otherwise

$$\rho(u) = 1/\gcd(\{W_0(u,v), W_0(v,u) \mid v \in V_0\})$$

where $n = |V_0|$ and $V_0 = \{v_1, v_2, \ldots, v_n\}$.
(1) The weighted graph $G_3 = (V_3, E_3.W_3)$ can be constructed in the following way:

$$V_3 = \{C_1, C_2, \ldots, C_N\},$$

For each $i, j \in \{1, 2, \ldots, N\}$,

$$W_3(C_i, C_j) = \rho(u)\rho(v)W_0(u,v) \text{for any } u \in C_i, v \in C_j,$$

are well-defined.

(2) Let $z = (\phi, \rho) : G_0 \to G_3$, where ϕ is the canonical surjection from V_0 onto V_3. Then, z is a surjective morphism and $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$.

Proof) Let $i, j \in \{1, 2, \ldots, N\}$. We shall show that for any $u, u' \in C_i$ and $v, v' \in C_j$,

$$\rho(u)\rho(v)W_0(u,v) = \rho(u')\rho(v')W_0(u',v'), \qquad (2)$$

Before proving the equation (2), under the condition that $\phi_k(u) = \phi_k(u')$ and $\phi_k(v) = \phi_k(v')$ hold for $k = 1, 2$, we show the equation (2). First,

$$\begin{aligned}&\rho_k(u)\rho_k(v)W_0(u,v) = W_k(\phi_k(u), \phi_k(v)) \\ &= W_k(\phi_k(u'), \phi_k(v')) = \rho_k(u')\rho_k(v')W_0(u',v')\end{aligned} \qquad (3)$$

holds and especially considering the case of $v = v'$, we have

$$\begin{aligned}&\rho_k(u)W_0(u,v) = \rho_k(u')W_0(u',v), \text{ and similarly} \\ &\rho_k(u)W_0(v,u) = \rho_k(u')W_0(v,u').\end{aligned} \qquad (4)$$

Next the following equation (5) holds.

$$\begin{aligned}&\text{neither } u \text{ nor } v \text{ is 0-isolated} \Longrightarrow \\ &\rho(u)\rho(v)\rho_k(u')\rho_k(v') = \rho(u')\rho(v')\rho_k(u)\rho_k(v).\end{aligned} \qquad (5)$$

Indeed since u and u' are not 0-isolated, the greatest common divisors give the following equations.

$$\begin{aligned}&\rho(u)\rho_k(u') \\ &= \rho(u')\rho(u)\rho_k(u')\rho^{-1}(u') \\ &= \rho(u')\rho(u)\rho_k(u')\gcd(\{W_0(u',v), W_0(v,u') \mid v \in V_0\}) \\ &= \rho(u')\rho(u)\rho_k(u)\gcd(\{W_0(u,v), W_0(v,u) \mid v \in V_0\}) \because (4) \\ &= \rho(u')\rho(u)\rho_k(u)\rho^{-1}(u) \\ &= \rho(u')\rho_k(u)\{\rho(u)\rho^{-1}(u)\} \\ &= \rho(u')\rho_k(u)\end{aligned}$$

Similarly we have $\rho(v)\rho_k(v') = \rho(v')\rho_k(v)$. These imply that the equation (5) holds. The equation (3)implies that $W_0(u,v) = 0 \iff W_0(u',v') = 0$. Since it is trivial in case of $W_0(u,v) = 0$, we may assume that $W_0(u,v) \ne 0$ and thus u is not 0-isolated.

$$\begin{aligned}&\rho(u)\rho(v)W_0(u,v) \\ &= \rho(u)\rho(v)\rho_k(u)^{-1}\rho_k(v)^{-1}\rho_k(u)\rho_k(v)W_0(u,v) \\ &= \rho(u)\rho(v)\rho_k(u)^{-1}\rho_k(v)^{-1}\rho_k(u')\rho_k(v')W_0(u',v') \\ &= \rho(u')\rho(v')\rho_k(u)^{-1}\rho_k(v)^{-1}\rho_k(u)\rho_k(v)W_0(u',v') \because (5) \\ &= \rho(u')\rho(v')W_0(u',v')\end{aligned}$$

If $\phi_k(u) = \phi_k(u')$ and $\phi_k(v) = \phi_k(v')$ hold for $k = 1, 2$, we have shown the equation (2) and return to the proof of the equation (2) in case of $u, u' \in C_i$ and $v, v' \in C_j$.

Since $u, u' \in C_i$ and $v, v' \in C_j$, there exist sequences

$$\begin{aligned}&s_0 = u, s_1, \ldots, s_\ell = u', \\ &\quad \text{with}(s_{k1}, s_{k+1}) \in \ker\phi_1 \cup \ker\phi_2 (0 \le k \le \ell), \\ &t_0 = v, t_1, \ldots, t_m = v', \\ &\quad \text{with}(t_{k-1}, t_k) \in \ker\phi_1 \cup \ker\phi_2 (0 \le k \le m).\end{aligned}$$

Then,

$$\begin{aligned}&\rho(s_0)\rho(t_0)W_0(s_0,t_0) = \rho(s_1)\rho(t_0)W_0(s_1,t_0) = \ldots \\ &= \rho(s_\ell)\rho(t_0)W_0(s_\ell,t_0) = \rho(s_\ell)\rho(t_1)W_0(s_\ell,t_1) = \ldots \\ &= \rho(s_\ell)\rho(t_m)W_0(s_\ell,t_m)\end{aligned}$$

Therefore the equation (2) and thus W_3 are well-defined.
(2) Let $k \in \{1, 2\}$. By the statement (1) above, the following maps are well-defined.

$$\begin{aligned}&{\phi_k}' : V_k \to V_3, v \mapsto \phi(u) && \text{where } \phi_k(u) = v, \\ &{\rho_k}' : V_k \to Q(R), v \mapsto \rho(u)\rho_k(u)^{-1} && \text{where } \phi_k(u) = v.\end{aligned}$$

For any $v, t \in V_k$, there exists $u, s \in V_0$ such that $\phi_k(u) = v$ and $\phi_k(s) = t$, and thus we have

$$W_3({\phi_k}'(v), {\phi_k}'(t)) = W_3(\phi(u), \phi(s)) = \rho(u)\rho(s)W_0(u, s)$$
$$= \rho(u)\rho(s)\rho_k(u)^{-1}\rho_k(s)^{-1}W_k(\phi_k(u), \phi_k(s))$$
$$= {\rho_k}'(v){\rho_k}'(t)W_k(v, t).$$

Therefore $x' = ({\phi_1}', {\rho_1}') : G_1 \to G_3$ and $y' = ({\phi_2}', {\rho_2}') : G_2 \to G_3$ are w-homomorphisms. We can easily show that ${\phi_k}'(k = 1, 2)$ are surjective, that is, $z = xx' = yy'$ $(x', y' \in \mathcal{S})$. Therefore $z\mathcal{S}^1 \subset x\mathcal{S}^1 \cap y\mathcal{S}^1$.

Conversely, we show that for any $w \subset x\mathcal{S}^1 \cap y\mathcal{S}^1$ there exists $z' \in \mathcal{S}^1$ such that $w = zz'$.

If we can write $w = xa = yb$, $a = (\psi_1, \sigma_1), v = (\psi_2, \sigma_2) \in \mathcal{S}$, then $w = (\psi, \sigma) = (\phi_1\psi_1, \rho_1 \otimes \phi_1\sigma_1) = (\phi_2\psi_2, \rho_2 \otimes \phi_2\sigma_2)$. Let $Im(w) = G_4 = (V_4, E_4, W_4)$

Let $u, u' \in C_i$. Since a sequence $s_0 = u, s_1, \ldots, s_\ell = u'$ such that for $0 \le j < \ell$ $\phi_1(s_j) = \phi_1(s_{j+1})$ or $\phi_2(s_j) = \phi_2(s_{j+1})$ exists, $\psi(s_j) = \psi(s_{j+1})$ holds. This implies that there exists $v \in V_4$ such that $C_i \subset \psi^{-1}(v)$. By Lemma 4, $w\mathcal{S}^1 \subset z\mathcal{S}^1$. Therefore, $x\mathcal{S}^1 \cap y\mathcal{S}^1 \subset z\mathcal{S}^1$. □

Corollary 2 (Diamond Property II)**.** Let $G_i = (V_i, E_i, W_i)$ $(i = 0, 1, 2)$ be weighted digraphs with $G_0 \sqsupseteq G_1$ and $G_0 \sqsupseteq G_2$. Then there exists a weighted digraoh G_3 such that $G_1 \sqsupseteq G_3$ and $G_2 \sqsupseteq G_3$.

We define the concept of irreducible forms of a Petri net with respect to $\sqsupseteq$.

Definition 5 (Irreducible)**.** *A weighted digraph G is called a $\sqsupseteq$-irreducible if $G \sqsupseteq G'$ implies $G \simeq G'$ for any weighted digraph G'. Then G is called an $\sqsupseteq$-irreducible form.* □

Corollary 3. Let G, G' and G'' be weighted digraphs with $G \sqsupseteq G'$ and $G \sqsupseteq G''$. Then one has: If G' and G'' are $\sqsupseteq$-irreducible, then $G' \simeq G''$.

Proof) Trivial by Corollary 2 and the definition of $\sqsupseteq$-irreducibility. □

4.Conclusion

In this paper we introduced our graph homomorphisms based on similarity of vertecies. Some algebraic properties related to them were investigeted. We first considered Green's relations and ideals in the semigroup $\mathcal{S}$ of all surjecvtive w-homomorphisms between two weighted digraphs, to which is adjoined the extra zero 0. In the semigroup $\mathcal{S}$, the intersection of principal left (resp. right) ideals is also a principal left (resp. right) ideal. This implies two kinds of diamond properties with respect to the preorder induced by surjective homomorphisms. It is technically interesting to construct such two kinds of synthesis of weighted digraphs.

The following problems remains open, for example, whether the intersection of two principal (two-sided) ideals is also a principal ideal in $\mathcal{S}$, whether weighted digraphs with the same irreducible form constitute a lattice with respect to the order up to isomorphism. In addition to these problems, we develope the application the elementary group theory to automorphim groups of weighted digraphs and would like to apply our graph homomorphism to formal languages and codes and to fundamental and common problems related to weighted digraphs.

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