A Linear Algorithm for Rectangular Drawings of Planar Graphs

LA-10

Md. Saidur Rahman,* Takao Nishizeki† and Shubhashis Ghosh‡

1 Introduction

A plane graph is a planar graph with a fixed embedding. A *rectangular drawing* of a plane graph G is a drawing of G in which each vertex is drawn as a point, each edge is drawn as a horizontal or vertical line segment without edge-crossings, and each face is drawn as a rectangle. (See Figure 1(a).) Not every plane graph has a rectangular drawing. We denote by Δ the *maximum degree* of G. If a plane graph G has a rectangular drawing, then $\Delta \leq 4$ and G must be biconnected and have four or more vertices of degree 2 on the outer face. Rahman *et al.* [RNN02] gave a necessary and sufficient condition for a plane graph of $\Delta \leq 3$ to have a rectangular drawing, and developed a linear-time algorithm to find a rectangular drawing of a plane graph if it exits.

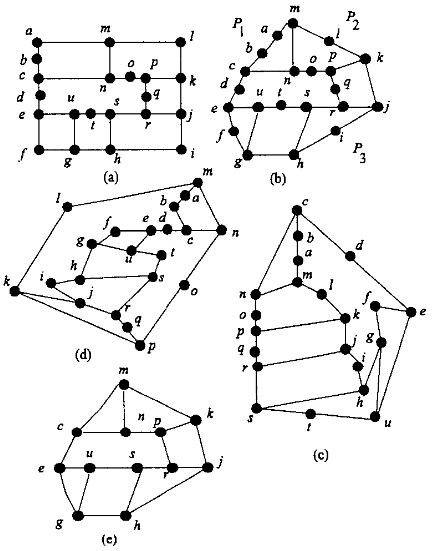

Figure 1: A rectangular drawing (a) and three different embeddings (b), (c) and (d) of the same graph which is subdivision of the graph in (e).

A planar graph G is said to have a rectangular drawing if at least one of the plane embeddings of G has a rectangular drawing. Figures 1(b), (c) and (d) depict three different plane embedings of the same planar graph. Among them only the embedding in Figure 1(b) has a rectangular drawing as illustrated in Figure 1(a). Thus the planar graph has a rectangular drawing. A rectangular drawing of a planar graph with $\Delta \leq 3$ has practical application in VLSI floorplanning [L90, RNN02] and architectural floorplanning [MKI00]. Examining whether a planar graph G of $\Delta \leq 3$ has a rectangular drawing is not a trivial problem, since G may have an exponential number of plane embeddings in general. A straightforward algorithm checking each of all the embeddings by the linear algorithm above in [RNN02] does not run in polynomial time. It has thus been desired to obtain more efficient algorithm.

We first consider "subdivisions" of planar 3-connected cubic graphs, and then consider general planar graphs of $\Delta \leq 3$. The plane graph in Figure 1(b) is a subdivision of the planar 3-connected cubic graph in Figure 1(e). A subdivision G of a planar 3-connected cubic graph has exactly one embedding for each face embedded as the outer face [NC88]. Hence G has an $O(n)$ number of embeddings, one for each chosen outer face. Thus, the straightforward algorithm takes time $O(n^2)$ to examine whether the planar graph G has a rectangular drawing. We then obtain a necessary and sufficient condition for a subdivision G of a planar 3-connected cubic graph to have a rectangular drawing, which leads to a linear-time algorithm to examine whether the planar graph G has a rectangular drawing. Using the algorithm, we finally give a linear-time algorithm to examine whether a general planar graph G of $\Delta \leq 3$ has a rectangular drawing and find a rectangular drawing of G if it exists.

2 Preliminaries

In this section we give some definitions and present preliminary results.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. We denote by $d(v)$ the *degree* of v. A graph G is called *cubic* if $d(v) = 3$ for every vertex v. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. We say that G is *k-connected* if $\kappa(G) \geq k$.

Let $P = w_0, w_1, w_2, ..., w_{k+1}$, $k \geq 1$, be a path of G such that $d(w_0) \geq 3, d(w_1) = d(w_2) = \cdots = d(w_k) = 2$, and $d(w_{k+1}) \geq 3$. Then we call the subpath $P' = w_1, w_2, ..., w_k$ of P a *chain* of G, and we call vertices w_0 and w_{k+1} the *supports* of the chain P'. Two chains on a cycle are *consecutive* if they have a common support.

Subdividing an edge (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0)$, w_1, w_2, ..., w_k, $v(= w_{k+1})$ through new vertices w_1, w_2, ..., w_k, $k \geq 1$, of degree 2. A graph G is said to be a *subdivision* of a graph G' if G is obtained from G' by subdividing some of the edges of G'.

A graph G is called *cyclically 4-edge-connected* if the removal of any three or fewer edges leaves a graph with only one connected component that has a cycle.

Let G be a planar biconnected graph, and let Γ be a plane embedding of G. The contour of a face is a cycle of G, and simply called a *face* or a *facial cycle*. We denote by $F_o(\Gamma)$

*Graduate School of Information Sciences, Tohoku University, Aoba-yama 05, Sendai 980-8579, Japan. Email: saidur@nishizeki.ecei.tohoku.ac.jp

†Graduate School of Information Sciences, Tohoku University, Aoba-yama 05, Sendai 980-8579, Japan. Email: nishi@ecei.tohoku.ac.jp, Fax number: +81-22-263-9301

‡Department of Computer Science, University of Alberta, Edmonton, Alberta T6G 2E8, Canada. Email: shubhash@cs.ulberta.ca

the outer face of Γ. For a cycle C of Γ, we call the plane subgraph of Γ inside C (including C) the *inner subgraph* $\Gamma_I(C)$ for C, and call the plane subgraph of Γ outside C (including C) the *outer subgraph* $\Gamma_O(C)$ for C. An edge of G which is incident to exactly one vertex of a cycle C and located outside C is called a *leg* of C. The vertex of C to which a leg is incident is called a *leg-vertex* of C. A cycle C in Γ is called a *k-legged cycle* of Γ if C has exactly k legs in Γ and there is no edge which joins two vertices on C and is located outside C. We call a face F of Γ a *peripheral face* for a 3-legged cycle C in Γ if F is in $\Gamma_O(C)$ and the contour of F contains some edges on C. Clearly there are exactly three peripheral faces for any 3-legged cycle in Γ.

A k-legged cycle C is called a *minimal k-legged cycle* if $G_I(C)$ does not contain any other k-legged cycle of G. We say that cycles C and C' in Γ are *independent* if $\Gamma_I(C)$ and $\Gamma_I(C')$ have no common vertex. A set $\mathcal{S}$ of cycles is *independent* if any pair of cycles in $\mathcal{S}$ are independent. A cycle C in a plane embedding Γ of G is called *regular* if the plane graph $\Gamma - \Gamma_I(C)$ has a cycle.

3 Subdivisions of Planar 3-connected Cubic Graphs

In this section we give a necessary and sufficient condition for a subdivision G of a planar 3-connected cubic graph to have a rectangular drawing, as in the following theorem.

Theorem 3.1 *Let G be a subdivision of a planar 3-connected cubic graph, and let Γ be an arbitrary plane embedding of G.*
(a) *Suppose first that G is cyclically 4-edge-connected, that is, Γ has no regular 3-legged cycle. Then the planar graph G has a rectangular drawing if and only if Γ has a face F such that (i) F contains at least four vertices of degree 2; (ii) there are at least two chains on F; and (iii) if there are exactly two chains on F, then they are not consecutive and each of them contains at least two vertices.*
(b) *Suppose next that G is not cyclically 4-edge connected, that is, Γ has a regular 3-legged cycle C. Let F_1, F_2 and F_3 be the three peripheral faces for C, and let Γ_1, Γ_2 and Γ_3 be the plane embeddings of G taking F_1, F_2 and F_3, respectively, as the outer face. Then the planar graph G has a rectangular drawing if and only if at least one of the three embeddings Γ_1, Γ_2 and Γ_3 has a rectangular drawing.*

Based on the characterization above we can give an algorithm, which we call Algorithm **Planar-Rectangular-Draw**, to find a rectangular drawing of G if it exits. We now have the following theorem.

Theorem 3.2 *Algorithm* **Planar-Rectangular-Draw** *examines in linear time whether a subdivision G of a planar 3-connected cubic graph has a rectangular drawing, and finds a rectangular drawing of G in linear time if it exists.*

4 Planar Graphs of $\Delta \leq 3$

In this section we give a linear-time algorithm to examine whether a general planar graph G of $\Delta \leq 3$ has a rectangular drawing and to find a rectangular drawing of G if it exists.

Let Γ be any arbitrary plane embedding of G. It is trivial to examine whether G has a rectangular drawing if Γ has one or two inner faces. We may thus assume that Γ has three or more inner faces.

If Γ has no regular 2-legged cycle, then G is a subdivision of a 3-connected cubic graph, and hence using the algorithm in Section 3, we can examine in linear time whether G has a rectangular drawing and can find a rectangular drawing of G if it exists. We may thus assume that Γ has a regular 2-legged cycle.

Let $C_1, C_2, \cdots, C_l$ be the regular 2-legged cycles in Γ, and let x_i, y_i, $1 \leq i \leq l$, be the two leg-vertices of C_i. Clearly $l = O(n)$ if G has n vertices. If the planar graph G has a rectangular drawing, then a plane embedding Γ^* of G has a rectangular drawing. The outer face $F_o(\Gamma^*)$ must contain all vertices $x_1, y_1, x_2, y_2, \cdots, x_l, y_l$; otherwise, Γ^* would not have a rectangular drawing as known from [RNN02]. Construct a graph G^+ from G by adding a dummy vertex z and dummy edges (x_i, z) and (y_i, z) for all indices i, $1 \leq i \leq l$. If G^+ is not planar, then G has no rectangular drawing. We thus assume that G^+ is planar and has an embedding Γ^+ such that z is embedded on the outer face.

We delete from Γ^+ the dummy vertex z and all dummy edges incident to z, and let Γ^* be the resulting plane embedding of G in which $F_o(\Gamma^*)$ contains all vertices $x_1, y_1, x_2, y_2, \cdots, x_l, y_l$. If Γ^* has three or more independent 2-legged cycles, then any plane embedding Γ' of G whose outer face contains all vertices $x_1, y_1, x_2, y_2, \cdots, x_l, y_l$ has three or more independent 2-legged cycles, and hence from [RNN02] Γ' has no rectangular drawing, and consequently the planar graph G has no rectangular drawing. We may thus assume that Γ^* has two or less independent 2-legged cycle.

Since Γ has a regular 2-legged cycle, Γ^* has two or more independent 2-legged cycles. Thus Γ^* has exactly two independent 2-legged cycles C_1 and C_2. We may assume without loss of generality that C_1 and C_2 are minimal 2-legged cycles. If we flip $\Gamma_I^*(C)$ for any 2-legged cycle C other than C_1 and C_2, then the outer face of the resulting embedding does not contain the leg-vertices of C_1 or C_2. By flipping $\Gamma_I^*(C_1)$ or $\Gamma_I^*(C_2)$ around the leg-vertices of C_1 or C_2, we have four different embeddings $\Gamma_1 (= \Gamma^*)$, Γ_2, Γ_3 and Γ_4 such that each $F_o(\Gamma_i)$, $1 \leq i \leq 4$, contains all vertices $x_1, y_1, x_2, y_2, \cdots, x_l, y_l$. Clearly, only these four embeddings Γ_1, Γ_2, Γ_3 and Γ_4 have all vertices $x_1, y_1, x_2, y_2, \cdots, x_l, y_l$ on the outer face. Thus G has a rectangular drawing if and only if any of Γ_1, Γ_2, Γ_3 and Γ_4 has a rectangular drawing.

Since the algorithm above takes linear time, we now have the following theorem.

Theorem 4.1 *Let G be a planar graph of $\Delta \leq 3$. Then one can examine in linear time whether G has a rectangular drawing and find a rectangular drawing of G if it exists.*

References

[L90] T. Lengauer, *Combinatirial Algorithms for Integrated Circuit Layout*, John Wiley & Sons, Chichester, 1990.

[MKI00] S. Munemoto, N. Katoh and G. Imamura, *Finding an optimal floor layout based on an orthogonal graph drawing algorithm*, J. Archit. Plann. Enviroment Eng. AIJ, No. 524, pp. 279-286, 2000.

[NC88] T. Nishizeki and N. Chiba, *Planar Graphs: Theory and Algorithms*, North-Holland, Amsterdam, 1988.

[RNN02] M. S. Rahman, S. Nakano and T. Nishizeki, *Rectangular drawings of plane graphs without designated corners*, Comp. Geom. Theo. Appl., 21(3), pp. 121-138, 2002.