Rectangle-of-Influence Drawings of Four-Connected Plane Graphs

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1 Introduction

Recently automatic aesthetic drawing of graphs have created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out. In this paper, we deal with the "rectangle-of-influence drawing" of a plane graph [BBM99]. Throughout the paper we denote by n the number of vertices of a graph G. The $W \times H$ *integer grid* consists of $W+1$ vertical grid lines and $H+1$ horizontal grid lines, and has a rectangular contour. W and H are called the *width* and *height* of the integer grid, respectively.

The most typical drawing of a plane graph G is the *straight-line drawing* in which all vertices of G are drawn as points and all edges are drawn as straight line segments without any edge-intersection. A straight-line drawing of G is called a *grid drawing* of G if all vertices of G are put on grid points of integer coordinates.

There are many results on grid drawings under additional constraints. For example, a grid drawing of a plane graph G is often pretty if every face boundary is drawn as a convex polygon. Such a drawing is called a *convex grid drawing* of G. Every 3-connected plane graph has a convex grid drawing on an $(n-2)\times(n-2)$ grid, and such a grid drawing can be found in linear time [CK97, ST92]. Figure 1(a) depicts a convex grid drawing of a plane graph obtained by the algorithm in [CK97]. On the other hand, a restricted class of graphs has a more compact convex grid drawing. For example, if G is a 4-connected plane graph and has at least four vertices on its outer face, then G has a convex grid drawing on a $W \times H$ grid such that $W+H \le n-1$, and one can find such a convex grid drawing in linear time [MNN2000]. Figure 1(b) depicts a convex grid drawing of the same graph obtained by the algorithm in [MNN2000].

In this paper, we deal with a type of gird drawings under another additional constraint, known as the *(open) rectangle-of-influence drawing*; it is a gird drawing such that there is no vertex in the proper inside of the axis-parallel rectangle defined by the two ends of any edge. A rectangle-of-influence drawing often looks pretty, since vertices are inclined to be separated from edges. The convex drawing in Fig. 1(a) is not a rectangle-of-influence drawing, while the convex drawing in Fig. 1(b) is a rectangle-of-influence drawing. A rectangle-of-influence drawing is called *closed* if the axis-parallel rectangle defined by the two ends of any edge contains no vertices except the ends on its boundary. Figure. 1(c) depicts a closed rectangle-of-influence drawing of the same plane graph as in Figs. 1(a) and (b).

Biedle *et al.* showed that a plane graph G has a (closed) rectangle-of-influence drawing on an $(n-1)\times(n-1)$ grid if there is no vertices in the interior of an any 3-cycle [BBM99]. The closed rectangle-of-influence drawing in Fig. 1(c) is obtained by their algorithm. Their result implies that any 4-connected plane graph with four or more vertices on the outer face has a (closed) rectangle-of-influence drawing on an $(n-1)\times(n-1)$ grid. However, the size of an integer grid required by a rectangle-of-influence drawing would be smaller than $(n-1)\times(n-1)$ for 4-connected plane graphs, but it has not been known how small the grid size is.

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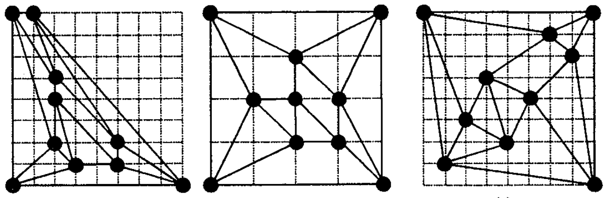

Figure 1: (a) Convex grid drawing, (b) open rectangle-of-influence drawing which is a convex drawing, and (c) closed rectangle-of-influence drawing of a plane graph.

In this paper we give an answer to this problem. That is, we show that the convex grid drawing of a 4-connected plane graph G found by the algorithm in [MNN2000] is always an (open) rectangle-of-influence drawing of G, and hence one can find in linear time a rectangle-of-influence grid drawing of G on a $W \times H$ grid such that $W+H \le n-1$ if G has n vertices. Since $W+H \le n-1$, the area $W \times H$ satisfies $W \times H \le \lceil (n-1)/2 \rceil \cdot \lfloor (n-1)/2 \rfloor$. The outer face boundary of G is always drawn as a rectangle. It should be noted that any 4-connected plane graph with four or more vertices on the outer face has a closed (and hence open) rectangle-of-influence drawing on an $(n-1)\times(n-1)$ grid [BBM99], but the drawing is not always a convex drawing as illustrated in Fig. 1(c).

2 Preliminaries

In this section we introduce some definitions.

Let $G=(V,E)$ be a simple connected undirected plane graph having no multiple edge or loop. V is the vertex set, and E is the edge set of G. Let $x(v)$ and $y(v)$ be the x- and y-coordinates of vertex $v \in V$, respectively. An edge joining vertices u and v is denoted by (u,v). The *degree* of a vertex v in G is the number of neighbors of v in G, and is denoted by $d(v,G)$.

A plane graph divides the plane into connected regions called *faces*. We denote the boundary of a face by a clockwise sequence of the vertices on the boundary. We call the boundary of the outer face of a plane graph G *the contour* of G, and denote it by $C_o(G)$.

The *open rectangle of an edge* is defined to be the interior of the rectangle defined by the ends of the edge. *An (open) rectangle-of-influence drawing of* G is a straight-line planar drawing of G such that there are no vertices in the open rectangle of any edge.

The "4-canonical decomposition" of a plane graph $G=(V,E)$ [NRN97] playing a crucial role in the algorithm in [MNN2000]. Let m be a natural number, and let $\Pi=(U_1,U_2,\cdots,U_m)$ be a partition of set V to m subsets $U_1,U_2,\cdots,U_m$ of V where $U_1 \bigcup U_2 \bigcup \cdots \bigcup U_m = V$ and $U_i \bigcap U_j = \phi$ for any i and j, $i \ne j$. Let G_k, $1 \le k \le m$, be the plane subgraph of G induced by the vertices in $U_1 \bigcup U_2 \bigcup \cdots \bigcup U_k$, and let $\overline{G_k}$ be the plane subgraph of G induced by the vertices in $U_{k+1} \bigcup U_{k+2} \bigcup \cdots \bigcup U_m$. Thus

$G = G_m = \overline{G_0}$. We say that Π is a *4-canonical decomposition* of G if the following three conditions are satisfied:

(co1) U_1 consists of the two ends of an edge on $C_o(G)$, and U_m consists of the two ends of another edge on $C_o(G)$;

(co2) for each k, $2 \le k \le m-1$, both G_k and $\overline{G_{k-1}}$ are biconnected and

(co3) for each k, $2 \le k \le m-1$, one of the following three conditions holds:

(a) U_k is a singleton set of a vertex u on $C_o(G_k)$ such that $d(u, G_k) \ge 2$ and $d(u, \overline{G_{k-1}}) \ge 2$.

(b) U_k is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) = 2$ and $d(u, \overline{G_{k-1}}) \ge 3$ for each vertex $u \in U_k$.

(c) U_k is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) \ge 3$ and $d(u, \overline{G_{k-1}}) = 2$ for each vertex $u \in U_k$.

We number all vertices of G by $1, 2, \cdots, n$ so that they appear in $U_1, U_2, \cdots, U_m$ in this order, and call each vertex in G by the number i, $1 \le i \le n$. Thus one can define an order $<$ among the vertices in G. The *lower neighbor* of u is the neighbors of u which are smaller than u. The *upper neighbor* of u is the neighbors of u which are larger than u.

3 Algorithm

In this section, we outline the algorithm in [MNN2000]. The algorithm first decides the x-coordinates of all vertices, and then decide the y-coordinates.

3.1 How to compute x-coordinates

The following **procedure** Construct-F constructs a directed forest $F = (V, E_F)$. All vertices in each component of F have the same x-coordinate; if there is a directed edge (i, j) in F, then $x(j) = x(i)$ and $y(j) > y(i)$.

Procedure Construct-F
begin $\{F = (V, E_F)\}$
$E_F := \phi$; {the initial forest $F = (V, \phi)$ consists of isolated vertices}
for $i := 1$ **to** n **do**
  if vertex i has upper neighbors j such that $d_{in}(j, F) = 0$ **then**
  let j be the largest one among them, and add a directed edge (i, j) to the directed graph F, that is, $E_F := E_F \bigcup \{(i, j)\}$;
end.

We then show how to arrange the paths in F from left to right. The algorithm decides a total order among all starting vertices of paths in F. For this purpose, using the following **procedure** Total-Order, the algorithm finds a directed path P going from vertex 1 to vertex 2 passing through all starting vertices of F.

Procedure Total-Order
begin
let P be the path directly going from vertex 1 to vertex 2;
for $i := 3$ **to** n **do**
  if $d_{in}(i, F) = 0$ **then** {i is a starting vertex of a path in F}
  begin
  let j be the first lower neighbor of i in the i's adjacency list in which the i's neighbors appear counterclockwise around i, and the first element of which is $w_m(i)$;
  let j' be the starting vertex of the path in F containing vertex j; $\{2 \ne j' < i\}$
  let k be the successor of j' in path P; {the path starting from vertex k in F has been put next to the right of the path starting from vertex j'}
  insert i in P between j' and k; {the path starting from i in F is put between the path starting from j' and the path starting from k}
  end
end.

3.2 How to compute y-coordinates

We now outline how to compute y-coordinates. For each k, $1 \le k \le m$, y-coordinates of all vertices in $U_k = \{u_1, u_2, \cdots, u_h\}$ are decided as the same integer, which is denoted by $y(U_k)$. Thus the path $u_1, u_2, \cdots, u_h$ on $C_o(G_k)$ is drawn as a horizontal line segment connecting points $(x(u_1), y(U_k))$ and $(x(u_h), y(U_k))$. Furthermore, the algorithm decides the y-coordinates $y(U_1), y(U_2), \cdots, y(U_m)$ in this order. Thus $H = y(U_m)$.

The algorithm first decides the y-coordinate $y(U_1)$ of $U_1 = \{1, 2\}$ as $y(U_1) = 0$. Thus it draws $G_1 = K_2$ as a horizontal line segment connecting points $(x(1), 0)$ and $(x(2), 0)$. The algorithm decides $y(U_k)$ to be either y_{max} or $y_{max} + 1$ so that the height H of the drawing becomes as small as possible. We omit the algorithm in this paper due to the page limitation.

4 Main Theorem

In this section, we prove that the convex drawing of a 4-connected plane graph G found by the algorithm in [MNN2000] is an open rectangle-of-influence drawing of G.

One can show that, each face of the convex drawing found by the algorithm in [MNN2000] is a particular convex polygon called an "trimmed rectangle." We call a polygon a *trimmed rectangle* if it can be obtained from an axis-parallel rectangle by trimming off some of the four corners.

Lemma 4.1 In the convex drawing of G found by the algorithm in [MNN2000], every face boundary is drawn as a trimmed rectangle, and each of the oblique sides is exactly one edge of G.

We immediately have the following theorem from Lemma 4.1.

Theorem 1 The convex drawing found by the algorithm in [MNN2000] is an open rectangle-of-influence drawing of G.

References

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