A-027

Convex Drawings of Internally Triconnected Plane Graphs on $O(n^2)$ Grids*

Xiao Zhou Takao Nishizeki

Abstract

In a convex grid drawing of a plane graph, every edge is drawn as a straight-line segment without any edge-intersection, every vertex is located at a grid point, and every facial cycle is drawn as a convex polygon. A plane graph G has a convex drawing if and only if G is internally triconnected. It has been known that an internally triconnected plane graph G of n vertices has a convex grid drawing on a grid of $O(n^3)$ area if the triconnected component decomposition tree of G has at most four leaves. In this paper, we improve the area bound $O(n^3)$ to $O(n^2)$, which is optimal within a coefficient. More precisely, we show that G has a convex grid drawing on a $2n \times 4n$ grid. We also present an algorithm to find such a drawing in linear time.

1 Introduction

Recently automatic aesthetic drawing of graphs has created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out [9]. The most typical drawing of a plane graph is a *straight line drawing*, in which all edges are drawn as straight line segments without any edge-intersection. A straight line drawing is called a *convex drawing* if every facial cycle is drawn as a convex polygon. One can find a convex drawing of a plane graph G in linear time if G has one [3, 4, 9].

A straight line drawing of a plane graph is called a *grid drawing* if all vertices are put on grid points of integer coordinates. This paper deals with a *convex grid drawing* of a plane graph. Throughout the paper we assume for simplicity that every vertex of a plane graph G has degree three or more. Then G has a convex drawing if and only if G is "internally triconnected" [2, 6, 7]. One may thus assume that G is internally triconnected. If either G is triconnected [1, 2] or the "triconnected component decomposition tree" $T(G)$ of G has two or three leaves [6], then G has a convex grid drawing on an $(n-1) \times (n-1)$ grid, where n is the number of vertices in G. If $T(G)$ has exactly four leaves, then G has a convex grid drawing on a $2n \times n^2$ grid [8]. Thus, G has a convex grid drawing of $O(n^3)$ area if $T(G)$ has at most four leaves.

In this paper, we improve the area bound $O(n^3)$ above to $O(n^2)$, which is optimal within a coefficent because a plane graph of nested triangles needs an $\Omega(n^2)$ area in any straight line drawing [5]. More precisely, we show that an internally triconnected plane graph G

*Graduate School of Information Sciences, Tohoku University, Aoba-yama 6-6-05, Sendai, 980-8579, Japan. This work is supported in part by JSPS, 19500001.

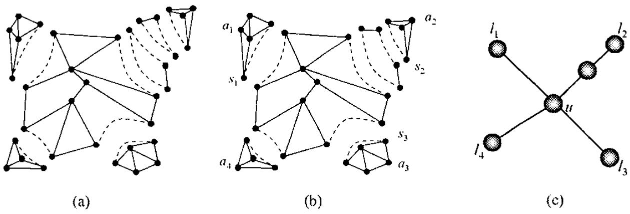

Figure 1: (a) Split components of the graph G in Fig. 2(a), (b) triconnected components of G, and (c) a decomposition tree $T(G)$.

has a convex grid drawing on a $2n \times 4n = O(n^2)$ grid if $T(G)$ has exactly four leaves, and present an algorithm to find such a drawing in linear time.

2 Outline of our algorithm

In this section, we outline our algorithm, which is a modification of the algorithm in [8].

The plane graph G in Fig. 2(a) is internally triconnected, the triconnected components of G are depicted in Fig. 1(b), and the triconnected component decomposition tree $T(G)$ of G having four leaves l_1, l_2, l_3 and l_4 is depicted in Fig. 1(c). We draw G so that the contour of the outer face of G is a rectangle, as illustrated in Fig. 2(e). We first appropriately choose four vertices a_1, a_2, a_3 and a_4 as the four apices of the rectangular contour. We then divide G into an upper subgraph G_u and a lower subgraph G_d, as illustrated in Fig. 2(b), so that G_u contains a_1 and a_2 and G_d contains a_3 and a_4. Using the "pentagon algorithm" in [8], we then obtain "inner convex" grid drawings D_u of G_u and D_d of G_d, both of $O(n^2)$ area, as illustrated in Figs. 2(c) and (d). More precisely, D_u has width $W(D_u) \leq 2n_u - 2$ and height $H(D_u) \leq 2n_u - 2$, and D_d has width $W(D_d) \leq 2n_d - 2$ and height $H(D_d) \leq 2n_d - 2$, where n_u and n_d are the numbers of vertices in G_u and G_d, respectively, and hence $n_u + n_d = n$. We then shift either vertex a_1 to the left or a_3 to the right so that these two drawings have the same width $\max\{2n_d - 2, 2n_u - 2\}$. We next arrange D_d and D_u so that $y(a_3) = y(a_4) = 0$ and $y(a_1) = y(a_2) = H(D_d) + H(D_u) + \max\{2n_d - 2, 2n_u - 2\} + 1$, as illustrated in Fig. 2(e), where $y(a_1), y(a_2), y(a_3)$ and $y(a_4)$ are the y-coordinates of a_1, a_2, a_3 and a_4, respectively. We finally draw, by straight line segments, all the edges of G that are contained in neither G_u nor G_d. Thus, the width $W(D)$ of the resulting drawing D of G is

$$W(D) \leq \max\{2n_d - 2, 2n_u - 2\} < 2n,$$

and the height $H(D)$ of D is

$$H(D) \leq 2n_d - 2 + 2n_u - 2 + \max\{2n_d - 2, 2n_u - 2\} + 1 < 4n.$$

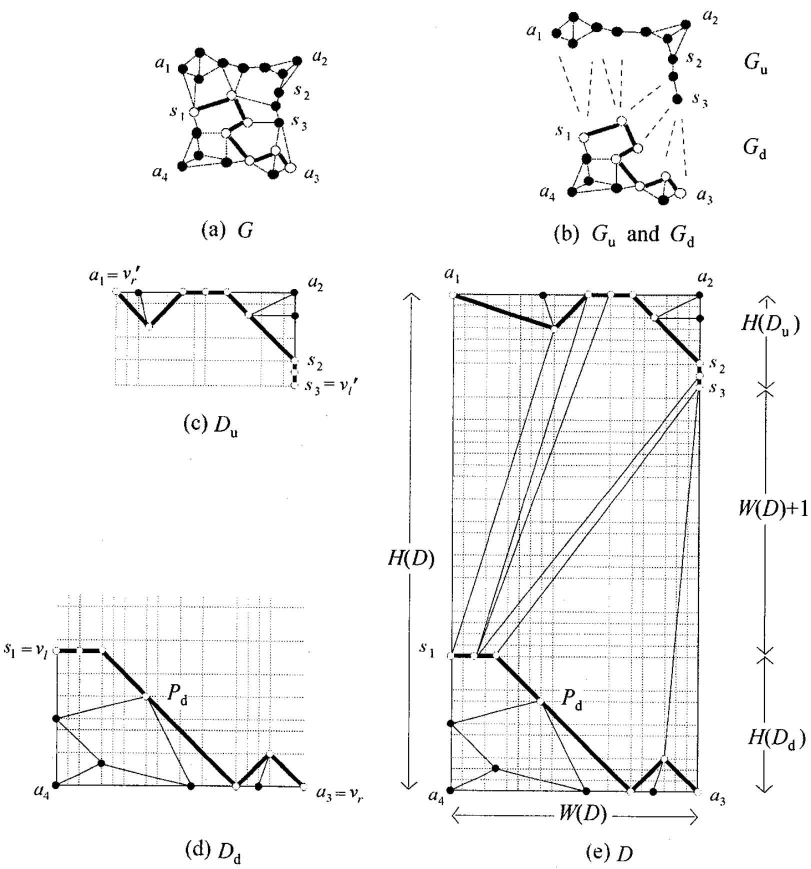

Figure 2: (a) A plane graph G, (b) subgraphs G_u and G_d, (c) a drawing D_u of G_u, (d) a drawing D_d of G_d, and (e) a convex grid drawing D of G.

Hence, the area of the drawing D is $2n \times 4n = O(n^2)$. The selection of apices a_1, a_2, a_3 and a_4, the division of G to G_u and G_d and some others are different from those in [8].

We thus have the following theorem.

Theorem 2.1 *Assume that G is an internally triconnected plane graph, every vertex of G has degree three or more, and the triconnected component decomposition tree $T(G)$ has exactly four leaves. Then our algorithm finds a convex grid drawing of G on a $2n \times 4n$ grid in linear time.*

The remaining problem is to obtain an algorithm for an internally triconnected plane graph whose decomposition tree has five or more leaves.

References

[1] N. Bonichon, S. Felsner and M. Mosbah, *Convex drawings of 3-connected plane graphs*, Algorithmica, 47, 4, pp. 399–420, 2007.

[2] M. Chrobak and G. Kant, *Convex grid drawings of 3-connected planar graphs*, International Journal of Computational Geometry and Applications, 7, pp. 211–223, 1997.

[3] N. Chiba, K. Onoguchi and T. Nishizeki, *Drawing planar graphs nicely*, Acta Inform., 22, pp. 187–201, 1985.

[4] N. Chiba, T. Yamanouchi and T. Nishizeki, *Linear algorithms for convex drawings of planar graphs*, in Progress in Graph Theory, J. A. Bondy and U. S. R. Murty (Eds.), Academic Press, pp. 153–173, 1984.

[5] D. Dolev, F. T. Leighton and H. Trickey, *Planar embedding of planar graphs*, Adcances in Computer Research - Vol 2, VLSI Theory, pp. 147–161, 1984.

[6] K. Miura, M. Azuma and T. Nishizeki, *Canonical decomposition, realizer, Schnyder labeling and orderly spanning trees of plane graphs*, International Journal of Foundations of Computer Science, 16, 1, pp. 117–141, 2005.

[7] K. Miura, M. Azuma and T. Nishizeki, *Convex drawings of plane graphs of minimum outer apices*, International Journal of Foundations of Computer Science, 17, 5, pp. 1115–1127, 2006.

[8] K. Miura, A. Kamada and T. Nishizeki, *Convex grid drawings of plane graphs of rectangular contours*, J. Graph Algorithms and Applications, 12, 2, pp. 197–224, 2008.

[9] T. Nishizeki and M. S. Rahman, *Planar Graph Drawing*, World Scientific, Singapore, 2004.