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An Extension of E-overlapping Notion in Term Rewriting Systems and its Applications

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Abstract

For nonlinear term rewriting systems(TRSs), the notion of E-overlapping extending that of usual overlapping has been proposed and some sufficient conditions for ensuring the decidability of some decision problems have been obtained by using this notion. Here, TRS R is E-overlapping if there exist two rewrite rules $\alpha \to \beta$ and $\alpha' \to \beta'$ in R such that α and some subterm of α' are unifiable modulo rewrite rules. In this paper, we introduce a new notion called LR-E-overlapping which is extending that of E-overlapping. Here, TRS R is LR-E-overlapping if there exist two rules $\alpha \to \beta$ and $\alpha' \to \beta'$ in R such that α or β is unifiable to some subterm of α' or β' modulo rewrite rules. Using this notion we give some new sufficient conditions for ensuring the decidability of some decision problems such as Church-Rosser property and E-unification for subclasses of nonlinear TRSs.

1 Introduction

A term rewriting system(TRS) is a set of directed equations called rewrite rules. The Church-Rosser(CR) is one of the most important property for TRSs in various applications and have received much attention so far. Here, a TRS is CR if every two interconvertible terms reduce to some common term by applications of rewrite rules. Although the CR property is undecidable in general, many sufficient conditions for ensuring the property have been obtained. However, only a few result have been obtained for nonlinear and nonterminating TRSs. The notion of E-overlapping was introduced by extending that of overlapping in [6]. TRS R is E-overlapping if there exist two rewrite rules $\alpha \to \beta$ and $\alpha' \to \beta'$ in R such that α and some subterm of α' are unifiable modulo rewrite rules. Using this notion, we have shown that non-E-overlapping TRSs are CR for some subclasses of nonlinear and nonterminating TRSs such as strongly depth-preserving TRSs [7, 2]. Here, TRS R is strongly depth-preserving if for each rule $\alpha \to \beta$ in R and any variable x appearing in both α and β, the minimal depth of x in α is greater than or equal to the maximal depth of x in β.

In this paper, we introduce a new notion called LR-E-overlapping which extends that of E-overlapping. Here, TRS R is LR-E-overlapping if there exist two rules $\alpha \to \beta$ and $\alpha' \to \beta'$ in R such that α or β is unifiable to some subterm of α' or β' modulo rewrite rules. Using this notion we give some new sufficient conditions for ensuring CR property for subclasses of nonlinear TRSs. Moreover, we use this notion to decide the E-unification problem. The E-unification problem for TRS is the problem of deciding, for TRS R and two terms s and t, whether s and t are unifiable by applications of rewrite rules in R. Althouge the E-unification problem is also undecidable in general, it has been shown that it is decidable for confluent(CR) semi-constructor TRSs [5]. Here, a semi-constructor TRS is such a TRS that all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms. Using this decidability result and the notion of LR-E-overlapping, we give a new decidability result of the E-unification problem for subclasses of nonlinear TRSs.

2 Preliminaries

The following definitions and notations are similar to those in [1, 8].

Let X be a set of variables, F be a finite set function symbols and $T(X, F)$ be the set of terms constructed from X and F. We use x, y, z as variables, c, d as constant symbols, f, g as function symbols of non-zero arity, and r, s, t as terms. A term is *ground* if it has no variable. Let G be the set of ground terms. For a term s, let $\mathsf{V}(s)$ be the set of variables occurring in s. The *root symbol* is defined as $\mathrm{root}(a) = a$ if a is a variable and

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$\mathsf{root}(f(t_1,\ldots,t_n)) = f$. A position in a term is expressed by a sequence of positive integers. Let $\mathcal{O}(s)$ be the set of positions of s. We use u, v as positions. Positions are partially ordered by the prefix ordering $\leq$. $s_{|u}$ denote the subterm of s at position u. The depth of position $u \in \mathcal{O}(s)$ is defined by $|u|$. The set of all minimal positions (w.r.t. $\leq$) of W is denoted by $\mathsf{Min}(W)$.

A *rewrite rule* is defined as a directed equation $\alpha \to \beta$ such that $\alpha \notin X$ and $\mathsf{V}(\alpha) \supseteq \mathsf{V}(\beta)$. A *TRS* R is a finite set of rewrite rules. We write $s \xrightarrow{u}_R t$ when there exist r, a substitution σ and $\alpha \to \beta \in R$ that satisfy $s = r[\alpha\sigma]_u$ and $t = r[\beta\sigma]_u$. In this case u is called the *redex position.* If u and R are clear from the context, we can drop them. Let $\leftarrow$ be the inverse of $\to$, $\leftrightarrow = \to \cup \leftarrow$, $\leftrightarrow^{=} = \leftrightarrow \cup =$ and $\downarrow = \to^* \cdot \leftarrow^*$. Let γ: $s_1 \overset{u_1}{\leftrightarrow} s_2 \cdots \overset{u_{n-1}}{\leftrightarrow} s_n$ be a *rewrite sequence.* This sequence is abbreviated to γ: $s_1 \leftrightarrow^* s_n$ and $\mathcal{R}(\gamma) = \{u_1, \cdots, u_{n-1}\}$ is the set of the redex positions of γ. If $u > \varepsilon$ for all $u \in \mathcal{R}(\gamma)$, then γ is called ε*-invariant* and we write γ: $s_1 \overset{>\varepsilon}{\leftrightarrow^*} s_n$.

For any set $\Delta \subseteq X \cup F$, let $\mathcal{O}_\Delta(s) = \{u \in \mathcal{O}(s) \mid \mathsf{root}(s_{|u}) \in \Delta\}$. Let $\mathcal{O}_x(s) = \mathcal{O}_{\{x\}}(s)$. The set D_R of *defined symbols* for a TRS R is defined as $D_R = \{\mathsf{root}(\alpha) \mid \alpha \to \beta \in R\}$. A term s is *semi-constructor* if for every subterm t of s,t is ground or $\mathsf{root}(t)$ is not a defined symbol.

Definition 2.1 A rule $\alpha \to \beta$ is *strongly depth-preserving* if for every $x \in \mathsf{V}(\beta)$, $\max\{|v| \mid v \in \mathcal{O}_x(\beta)\} \leq \min\{|u| \mid u \in \mathcal{O}_x(\alpha)\}$ holds. A TRS R is *strongly depth-preserving* if every rule in R is strongly depth-preserving. A rule $\alpha \to \beta$ is *semi-constructor* if β is semi-constructor. A TRS R is *semi-constructor* if every rule in R is semi-constructor. A TRS R is *confluent* or *Church-Rosser(CR)* if $\leftrightarrow^*_R = \downarrow_R$.

Definition 2.2 Terms s and t are E-overlapping at $u(\in \mathcal{O}(s) \setminus \mathcal{O}_X(s))$ if there exist substitutions θ, θ' such that $s_{|u}\theta \overset{>\varepsilon}{\leftrightarrow^*} t\theta'$. Terms s and t are ω-overlapping at $u(\in \mathcal{O}(s) \setminus \mathcal{O}_X(s))$ if there exist substitutions θ, θ' such that $s_{|u}\theta = t\theta'$, where θ and θ' may be cyclic. A TRS R is E-overlapping(ω-overlapping) if there exist $\alpha \to \beta, \alpha' \to \beta' \in R$ and $u \in \mathcal{O}(\alpha) \setminus \mathcal{O}_X(\alpha)$ such that α and α' are E-overlapping(ω-overlapping) at u, where $(\alpha \to \beta) = (\alpha' \to \beta')$ implies $u \neq \varepsilon$.

3 Fundamental results and new E-overlapping notion

Let R be a TRS over $T(X,F)$ and $R' = \{\alpha_i \to \beta_i \mid 1 \leq i \leq n\} \subseteq \{\alpha \to \beta \in R \mid \beta \notin X\}$. Let $F' = \{f_1, \cdots, f_n\}$ where $F \cap F' = \emptyset$. We define a TRS $\Phi(R,R')$ over $T(X, F \cup F')$ as follows:

$$\Phi(R,R') = R \setminus R' \cup \{\alpha_i \to f_i(x_1,\cdots,x_k), \beta_i \to f_i(x_1,\cdots,x_k) \mid \mathsf{V}(\beta_i) = \{x_1,\cdots,x_k\}, 1 \leq i \leq n\}$$

Definition 3.1 We define a mapping $\phi_{R'} : T(X, F \cup F') \to T(X,F)$ as follows.

$$\phi_{R'}(t) = \begin{cases} \beta_i\sigma_i & (\text{if } t = f_i(t_1,\cdots,t_k), f_i \in F') \\ f(\phi_{R'}(t_1),\cdots,\phi_{R'}(t_k)) & (\text{if } t = f(t_1,\cdots,t_k), f \in F) \\ t & (\text{if } t \in X) \end{cases}$$

Here, $\beta_i \to f_i(x_1,\cdots,x_k) \in R'$ and $\sigma_i = \{x_j \to \phi_{R'}(t_j) \mid 1 \leq j \leq k\}$.

For TRSs R and $\Phi(R,R')$, the following lemmata hold.

Lemma 3.2 If $s \to_{\Phi(R,R')} t$ then $\phi_{R'}(s) \to^*_R \phi_{R'}(t)$ for every $s,t \in T(X, F \cup F')$.
Proof By induction on the structure of s.
Basis: Since $s \in X$, $s \to_{\Phi(R,R')} t$ is impossible, so that this lemma holds.
Induction step: Let $s \xrightarrow{p}_{\Phi(R,R')} t$.

Case of $p > \varepsilon$: Let $s = f(s_1,\cdots,s_k)$, then $t = f(t_1,\cdots,t_k)$ and $s_j \to^{=}_{\Phi(R,R')} t_j$ for every $j \in \{1,\cdots,k\}$. By the induction hypothesis, $\phi_{R'}(s_j) \to^*_R \phi_{R'}(t_j)$ for every $j \in \{1,\cdots,k\}$. Thus, if $f \in F$ then $\phi_{R'}(s) \to^*_R \phi_{R'}(t)$ holds. Otherwise, since $f = f_i$ for some $\beta_i \to f_i(x_1,\cdots,x_k) \in \Phi(R,R')$, $\phi_{R'}(s) = \beta_i\sigma$ and $\phi_{R'}(t) = \beta_i\sigma'$ where $\sigma = \{x_j \to \phi_{R'}(s_j) \mid 1 \leq j \leq k\}$ and $\sigma' = \{x_j \to \phi_{R'}(t_j) \mid 1 \leq j \leq k\}$. Thus, $\phi_{R'}(s) \to^*_R \phi_{R'}(t)$ holds.

Case of $p = \varepsilon$: Let $s = \alpha\theta \to_{\Phi(R,R')} \beta\theta = t$ where $\alpha \to \beta$ is a rewrite rule. Obviously, $\alpha\theta' \to_{\Phi(R,R')} \beta\theta'$ holds for $\theta' = \{x \to \phi_{R'}(r) \mid x \to r \in \theta\}$. If $\alpha \to \beta \in R \setminus R'$ then $\phi_{R'}(s) = \alpha\theta' \to_R \beta\theta' = \phi_{R'}(t)$ holds. Otherwise, if $\alpha = \alpha_i$ for some $i \in \{1,\cdots,n\}$ then $\beta = f_i(t_1,\cdots,t_k)$, so that $\phi_{R'}(s) = \alpha_i\theta' \to_R \beta_i\theta' = \phi_{R'}(t)$ holds. If $\alpha = \beta_i$ for some $i \in \{1,\cdots,n\}$ then $\beta = f_i(t_1,\cdots,t_k)$, so that $\phi(s) = \beta_i\theta' = \phi(t)$ holds. □

Lemma 3.3 If $\Phi(R,R')$ is CR over $T(X,F\cup F')$ then R is CR over $T(X,F)$.
Proof Let $s \leftrightarrow_R^* t$ for some $s,t \in T(X,F)$, then $s \leftrightarrow_{\Phi(R,R')}^* t$ obviously holds. Since $\Phi(R,R')$ is CR over $T(X,F\cup F')$, there exists $r \in T(X,F\cup F')$ such that $s \rightarrow_{\Phi(R,R')}^* r$ and $t \rightarrow_{\Phi(R,R')}^* r$. By Lemma 3.2, $\phi_{R'}(s) \rightarrow_R^* \phi_{R'}(r)$ and $\phi_{R'}(t) \rightarrow_R^* \phi_{R'}(r)$ hold. By $s,t \in T(X,F)$, $\phi_{R'}(s)=s$ and $\phi_{R'}(t)=t$ hold. Thus, $s \downarrow_R t$ holds. □

Using these lemmata, we can define a new notion extending that of E-overlapping.

Definition 3.4 A TRS R is $\mathrm{LR}_{R'}$-E-overlapping if $\Phi(R,R')$ is E-overlapping.

4 Applications of the notion of LR-E-overlapping

In this section, we give some new sufficient conditions for ensuring the decidability of some decision problems such as Church-Rosser property and E-unification for subclasses of nonlinear TRSs using the notion of LR-E-overlapping.

4.1 CR Property

Let R be a TRS and $R_{\mathrm{nsdp}} = \{\alpha \to \beta \in R \mid \alpha \to \beta \text{ is non strongly depth-preserving}\}$. Obviously, $\Phi(R,R_{\mathrm{nsdp}})$ is strongly depth-preserving. Since strongly depth-preserving and non-E-overlapping TRSs are CR [2], the following theorem holds by Lemma 3.3.

Theorem 4.1 If R is non-$\mathrm{LR}_{R_{\mathrm{nsdp}}}$-E-overlapping (i.e., $\Phi(R,R_{\mathrm{nsdp}})$ is non-E-overlapping) then R is CR.

The root-E-closed property introduced in [3] is also a sufficient condition for ensuring CR property for strongly depth-preserving TRSs, which is a more general than non-E-overlapping [3]. Thus, the following corollary holds.

Corollary 4.2 If $\Phi(R,R_{\mathrm{nsdp}})$ is root-E-closed then R is CR.

Example 4.3 Let $R = \{\mathsf{c} \to \mathsf{g}(\mathsf{c},\mathsf{c}),\ \mathsf{g}(x,x) \to \mathsf{f}(x,\mathsf{g}(x,\mathsf{h}(x))),\ \mathsf{f}(x,x) \to \mathsf{a}\}$. Here, $R_{\mathrm{nsdp}} = \{\mathsf{g}(x,x) \to \mathsf{f}(x,\mathsf{g}(x,\mathsf{h}(x)))\}$ and $\Phi(R,R_{\mathrm{nsdp}}) = \{\mathsf{c} \to \mathsf{g}(\mathsf{c},\mathsf{c}),\ \mathsf{g}(x,x) \to \mathsf{f}_1(x),\ \mathsf{f}(x,\mathsf{g}(x,\mathsf{h}(x))) \to \mathsf{f}_1(x),\ \mathsf{f}(x,x) \to \mathsf{a}\}$. Since $\Phi(R,R_{\mathrm{nsdp}})$ is non-E-overlapping, we can ensure that R is CR.

Non-ω-overlapping property is a sufficient and decidable condition for ensuring non-E-overlapping property for strongly depth-preserving TRSs [4]. Thus, the following corollary holds.

Corollary 4.4 If $\Phi(R,R_{\mathrm{nsdp}})$ is non-ω-overlapping then R is CR.

Example 4.5 Let $R = \{\mathsf{c} \to \mathsf{g}(\mathsf{c},\mathsf{c}),\ \mathsf{g}(x,x) \to \mathsf{f}(\mathsf{l}(x),\mathsf{g}(\mathsf{l}(x),\mathsf{h}(x))),\ \mathsf{f}(x,x) \to \mathsf{a}\}$. Here, $R_{\mathrm{nsdp}} = \{\mathsf{g}(x,x) \to \mathsf{f}(\mathsf{l}(x),\mathsf{g}(\mathsf{l}(x),\mathsf{h}(x)))\}$ and $\Phi(R,R_{\mathrm{nsdp}}) = \{\mathsf{c} \to \mathsf{g}(\mathsf{c},\mathsf{c}),\ \mathsf{g}(x,x) \to \mathsf{f}_1(x),\ \mathsf{f}(\mathsf{l}(x),\mathsf{g}(\mathsf{l}(x),\mathsf{h}(x))) \to \mathsf{f}_1(x),\ \mathsf{f}(x,x) \to \mathsf{a}\}$. Since $\Phi(R,R_{\mathrm{nsdp}})$ is non-ω-overlapping, we can ensure that R is CR.

4.2 E-unification problem

Definition 4.6 Terms s and t are E-unifiable for TRS R if there exists a substitution θ such that $s\theta \leftrightarrow_R^* t\theta$.

Let R be a TRS over $T(X,F)$ and $R_{\mathrm{nsc}} = \{\alpha \to \beta \in R \mid \alpha \to \beta \text{ is non semi-constructor}\} = \{\alpha_i \to \beta_i \mid 1 \le i \le m\}$. For $\alpha_i \to \beta_i \in R_{\mathrm{nsc}}$, let $U_i = \mathsf{Min}\{u \in \mathcal{O}(\beta_i) \setminus \mathcal{O}_G(\beta_i) \mid \mathsf{root}(\beta_{i|u}) \in D_R\} = \{u_{i1},\cdots,u_{ik}\}$. Note that $U_i \neq \emptyset$. Let $F_i' = \{f_{i1},\cdots,f_{ik} \mid k = |U_i|\}$ and $F' = \bigcup_{1\le i\le n} F_i'$, where $F \cap F' = \emptyset$. TRS $\Psi(R)$ over $T(X,F\cup F')$ is constructed as follows:

$$\Psi(R) = R \setminus R_{\mathrm{nsc}} \cup \{\alpha_i \to \beta_i[t_{i1},\cdots,t_{ik}]_{(u_{i1},\cdots,u_{ik})}, \beta_{i|u_{ij}} \to t_{ij} \mid 1 \le j \le k\}$$

where $t_{ij} = f_{ij}(x_1,\cdots,x_l)$, $f_{ij} \in F_i'$ and $\mathsf{V}(\beta_{i|u_{ij}}) = \{x_1,\cdots,x_l\}$. Note that $F' \not\subseteq D_R = D_{\Psi(R)}$ so that $\Psi(R)$ is a semi-constructor TRS.

We define $\phi : T(X,F\cup F') \to T(X,F)$ as follows.

$$\phi(t) = \begin{cases} \beta_{i|u_{ij}}\sigma_{ij} & (\text{if } t = f_{ij}(t_1,\cdots,t_l), f_{ij} \in F') \\ f(\phi(t_1),\cdots,\phi(t_l)) & (\text{if } t = f(t_1,\cdots,t_l), f \in F) \\ t & (\text{if } t \in X) \end{cases}$$

Here, $\beta_{i|u_{ij}} \to f_{ij}(x_1,\cdots,x_l) \in \Psi(R)$ and $\sigma_{ij} = \{x_k \to \phi(t_k) \mid 1 \le k \le l\}$.

For TRSs R and $\Psi(R)$, the following lemmata hold.

Lemma 4.7 If $s \rightarrow_{\Psi(R)} t$ then $\phi(s) \rightarrow_R^* \phi(t)$ for every $s, t \in T(X, F \cup F')$.
Proof By induction on the structure of s.
Basis: Since $s \in X$, $s \rightarrow_{\Psi(R)} t$ is impossible, so that this lemma holds.
Induction step: Let $s \xrightarrow{p}_{\Psi(R)} t$.
Case of $p > \varepsilon$: Similar to Lemma 3.2.
Case of $p = \varepsilon$: Let $s = \alpha\theta \rightarrow_{\Psi(R)} \beta\theta = t$ where $\alpha \rightarrow \beta$ is a rewrite rule. Obviously, $\alpha\theta' \rightarrow_{\Psi(R)} \beta\theta'$ holds for $\theta' = \{x \rightarrow \phi(r) \mid x \rightarrow r \in \theta\}$. If $\alpha \rightarrow \beta \in \Psi(R)$ then $\phi(s) = \alpha\theta' \rightarrow_R \beta\theta' = \phi(t)$ holds. Otherwise, if $\alpha = \alpha_i$ for some $i \in \{1, \cdots, n\}$ then $\beta = \beta_i[t_{i1}, \cdots, t_{ik}]_{(u_{i1}, \cdots, u_{ik})}$, so that $\phi(s) = \alpha_i\theta' \rightarrow_R \beta_i\theta' = \phi(t)$ holds. If $\alpha = \beta_{i|u_{ij}}$ for some $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, k\}$ then $\beta = f_{ij}(x_1, \cdots, x_l)$, so that $\phi(s) = \beta_{i|u_{ij}}\theta' = \phi(t)$ holds. □

Lemma 4.8 For any $s, t \in T(X, F)$, s and t are E-unifiable for R iff s and t are E-unifiable for $\Psi(R)$.
Proof If part: By Lemma 4.7. Only if part: Obvious. □

In [3], the notion of strongly weight-preserving which extends that of strongly depth-preserving was introduced. We can easily show that every semi-constructor TRS is strongly weight-preserving. It is known that strongly weight-preserving and non-E-overlapping(or root-E-closed) TRSs are CR [3]. Moreover, the E-unification problem is decidable for confluent(CR) semi-constructor TRSs [5], so that we can deduce the following theorem by Lemma 4.8.

Theorem 4.9 If $\Psi(R)$ is non-E-overlapping(or root-E-closed) then the E-unification problem for R is decidable.

5 Conclusion

In this paper, we have introduced a new notion called LR-E-overlapping which extends that of E-overlapping. Using this new notion we have given some new sufficient conditions for ensuring some properties such as CR and the decidability of the E-unification problem for subclasses of nonlinear TRSs.

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