A-002

Balanced (C_4, C_{12})-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_{12} be *the 4-cycle* and *the 12-cycle*, respectively. *The* (C_4, C_{12})*-2t-foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_{12}'s with a common vertex and the common vertex is called *the center of the* (C_4, C_{12})*-2t-foil.* In particular, the (C_4, C_{12})-2-foil is called *the* (C_4, C_{12})*-bowtie.* When K_n is decomposed into edge-disjoint sum of (C_4, C_{12})-$2t$-foils, we say that K_n *has a* (C_4, C_{12})*-2t-foil decomposition.* Moreover, when every vertex of K_n appears in the same number of (C_4, C_{12})-$2t$-foils, we say that K_n *has a balanced* (C_4, C_{12})*-2t-foil decomposition* and this number is called *the replication number.*
Note that (C_4, C_{12})-$2t$-foil has $14t+1$ vertices and $16t$ edges.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*
In this sense, our balanced (C_4, C_{12})-$2t$-foil decomposition of K_n is to be known as *a balanced* (C_4, C_{12})*-2t-foil system.*

2. Balanced (C_4, C_{12})-$2t$-foil decomposition of K_n

Theorem. K_n has a balanced (C_4, C_{12})-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{32t}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_{12})-$2t$-foil decomposition. Let b be the number of (C_4, C_{12})-$2t$-foils and r be the replication number. Then $b = n(n-1)/32t$ and $r = (14t+1)(n-1)/32t$. Among r (C_4, C_{12})-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_{12})-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/32t$ and $r_2 = 14(n-1)/32$. Therefore, $n \equiv 1 \pmod{32t}$ is necessary.
(Sufficiency) Put $n = 32st+1$, $T = st$. Then $n = 32T+1$.
Construct a (C_4, C_{12})-$2T$-foil as follows:
$\{(32T+1, 2T+1, 9T+2, 3T+1), (32T+1, 1, 4T+2, 14T+2, 24T+3, 12T+2, 29T+3, 13T+2, 26T+3, 15T+2, 6T+2, T+1)\} \cup$
$\{(32T+1, 2T+2, 9T+4, 3T+2), (32T+1, 2, 4T+4, 14T+3, 24T+5, 12T+3, 29T+5, 13T+3, 26T+5, 15T+3, 6T+4, T+2)\} \cup$
$\{(32T+1, 2T+3, 9T+6, 3T+3), (32T+1, 3, 4T+6, 14T+4, 24T+7, 12T+4, 29T+7, 13T+4, 26T+7, 15T+4, 6T+6, T+3)\} \cup ... \cup$
$\{(32T+1, 3T, 11T, 4T), (32T+1, T, 6T, 15T+1, 26T+1, 13T+1, 31T+1, 14T+1, 28T+1, 16T+1, 8T, 2T)\}$.
($16T$ edges, $16T$ all lengths)
Decompose the (C_4, C_{12})-$2T$-foil into s (C_4, C_{12})-$2t$-foils. Then these s starters comprise a balanced (C_4, C_{12})-$2t$-foil decomposition of K_n.

Corollary. K_n has a balanced (C_4, C_{12})-bowtie decomposition if and only if $n \equiv 1 \pmod{32}$.

Example 1. A (C_4, C_{12})-2-foil of K_{33}.
$\{(33, 3, 11, 4), (33, 1, 6, 16, 27, 14, 32, 15, 29, 17, 8, 2)\}$.
(16 edges, 16 all lengths)
This starter comprises a balanced (C_4, C_{12})-2-foil decomposition of K_{33}.

Example 2. A (C_4, C_{12})-4-foil of K_{65}.
$\{(65, 5, 20, 7), (65, 1, 10, 30, 51, 26, 61, 28, 55, 32, 14, 3)\} \cup$
$\{(65, 6, 22, 8), (65, 2, 12, 31, 53, 27, 63, 29, 57, 33, 16, 4)\}$.
(32 edges, 32 all lengths)
This starter comprises a balanced (C_4, C_{12})-4-foil decomposition of K_{65}.

Example 3. A (C_4, C_{12})-6-foil of K_{97}.
$\{(97, 7, 29, 10), (97, 1, 14, 44, 75, 38, 90, 41, 81, 47, 20, 4)\}$
$\cup$
$\{(97, 8, 31, 11), (97, 2, 16, 45, 77, 39, 92, 42, 83, 48, 22, 5)\}$
$\cup$
$\{(97, 9, 33, 12), (97, 3, 18, 46, 79, 40, 94, 43, 85, 49, 24, 6)\}$.
(48 edges, 48 all lengths)
This starter comprises a balanced (C_4, C_{12})-6-foil decomposition of K_{97}.

Example 4. A (C_4, C_{12})-8-foil of K_{129}.
$\{(129,9,38,13),(129,1,18,58,99,50,119,54,107,62,26,5)\}$
$\cup$
$\{(129,10,40,14),(129,2,20,59,101,51,121,55,109,63,28,6)\}$
$\cup$
$\{(129,11,42,15),(129,3,22,60,103,52,123,56,111,64,30,7)\}$
$\cup$
$\{(129,12,44,16),(129,4,24,61,105,53,125,57,113,65,32,8)\}$.
(64 edges, 64 all lengths)
This starter comprises a balanced (C_4, C_{12})-8-foil decomposition of K_{129}.

Example 5. A (C_4, C_{12})-10-foil of K_{161}.
$\{(161,11,47,16),(161,1,22,72,123,62,148,67,133,77,32,6)\}$
$\cup$
$\{(161,12,49,17),(161,2,24,73,125,63,150,68,135,78,34,7)\}$
$\cup$
$\{(161,13,51,18),(161,3,26,74,127,64,152,69,137,79,36,8)\}$
$\cup$
$\{(161,14,53,19),(161,4,28,75,129,65,154,70,139,80,38,9)\}$
$\cup$
$\{(161,15,55,20),(161,5,30,76,131,66,156,71,141,81,40,10)\}$.
(80 edges, 80 all lengths)
This starter comprises a balanced (C_4, C_{12})-10-foil decomposition of K_{161}.

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