Degree-Constrained Orientation for Graphs with Polynomially Many Potentially Maximal Cliques*

Hirotaka Ono[1,a)] Yota Otachi[2,b)]

Abstract: A degree-constrained orientation of an undirected graph G is an assignment of a direction to each edge in G such that the number of vertices of outdegree at most (or at least) W is maximized. Finding such an orientation is NP-hard even if W is a fixed constant because this setting of problems contains well-known NP-hard problems such as Max Independent Set Problem. In this paper, we show that the problem of any constant W can be solved in polynomial time for graphs in the classes of chordal graphs, circular-arc graphs, d-trapezoid graphs, and chordal bipartite graphs. The idea is to utilize a dynamic programming algorithm framework proposed by Fomin, Todinca, and Villanger [SIAM J. Comput. 44 (2015) 57–87], but it is far from obvious to confirm that our problem and target graph classes are included in the framework. We show that pairs of our problem and many of the graph classes above satisfy the condition of application, which leads to the polynomial-time solvability.

Keywords: Graph orientation, Treewidth,

1. Introduction

Let $G = (V, E)$ be an undirected (multi-)graph. An *orientation of* G is a function that maps each undirected edge $\{u, v\}$ in E to one of the two possible directed edges (u, v) and (v, u). For any orientation Λ of G, define $\Lambda(E) = \bigcup_{e \in E}\{\Lambda(e)\}$ and let $\Lambda(G)$ denote the directed graph $(V, \Lambda(E))$. For any vertex $u \in V$, the *outdegree of* u *under* Λ is defined as $d^+_\Lambda(u) = |\{(u, v) : (u, v) \in \Lambda(E)\}|$, i.e., the number of outgoing edges from u in $\Lambda(G)$. For any non-negative integer W, a vertex $u \in V$ is called W*-light in* $\Lambda(G)$ if $d^+_\Lambda(u) \le W$, and W*-heavy in* $\Lambda(G)$ if $d^+_\Lambda(u) \ge W$. For any $U \subseteq V$, if all the vertices in U are W-light (resp., W-heavy), we say that U is W-light (resp., W-heavy).

For any fixed non-negative integer W, the following optimization problems (introduced in [3], see also [4]) are defined, where the input is an undirected (multi-)graph $G = (V, E)$ and a non-negative integer W:

- Max W-Light: Output an orientation Λ of G s.t. $\left|\{u \in V : d^+_\Lambda(u) \le W\}\right|$ is maximized.
- Max W-Heavy: Output an orientation Λ of G s.t. $\left|\{u \in V : d^+_\Lambda(u) \ge W\}\right|$ is maximized.

Symmetrically, we can consider the following problems:

- Min W-Light: Output an orientation Λ of G s.t. $\left|\{u \in V : d^+_\Lambda(u) \le W\}\right|$ is minimized.
- Min W-Heavy: Output an orientation Λ of G s.t. $\left|\{u \in V : d^+_\Lambda(u) \ge W\}\right|$ is minimized.

Observe that Max W-Light (resp., Max W-Heavy) and Min $(W+1)$-Heavy (resp., Min $(W-1)$-Light) are *supplementary problems* in the sense that an exact algorithm for one gives an exact algorithm for the other, though their approximability properties may differ. Since this paper focuses on the polynomial-time solvability, we consider only Max W-Light and Max W-Heavy.

Although these problems are recently introduced in [3], they contain several old and well-known NP-hard problems. For example, Max W-Light with $W = 0$ is equivalent to Maximum Independent Set Problem (thus Min W-Heavy with $W = 1$ is equivalent to Minimum Vertex Cover Problem). In fact, if a graph $G = (V, E)$ has an independent set $U \subseteq V$ (that is, no adjacent vertices are in U), all vertices in U are 0-light in $\Lambda(G)$, where $\Lambda(G)$ is defined as follows: if an edge $e = \{u, v\}$ satisfies $e \cap U = \{u\}$ (resp., $\{v\}$), let $\Lambda(e) = (v, u)$ (resp., $\Lambda(e) = (u, v)$), and otherwise assign e with an arbitrary direction. On the other hand, if a vertex set U of graph $G = (V, E)$ is 0-light in some $\Lambda(G)$, U forms an independent set.

For these problems, the same authors of [3] investigate the approximability [4]. They got comprehensive results on the approximability of the problems. Some of the results are listed as follows:

- For every fixed $W \ge 1$, Max W-Light cannot be approximated within $(n/W)^{1-\varepsilon}$ in polynomial time, unless P = NP. On the positive side, there exists a polynomial-time $(n/(2W + 1))$-approximation algorithm for Max W-Light.
- For every fixed $W \ge 1$, Min W-Heavy cannot be approximated within 1.3606 in polynomial time, unless P = NP. On the positive side, they show that Min W-Heavy can be approximated within a ratio of $\ln(W + 1) + 1$ in polynomial time for every fixed $W \ge 2$.
- Min W-Light can be approximated within $\ln(W + 1) + 1$ for

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1 Faculty of Economics, Kyushu University. 6-19-1, Hakozaki, Higashi-ku, Fukuoka 812–8581, Japan
2 School of Information Science, Japan Advanced Institute of Science and Technology. Asahidai 1-1, Nomi, Ishikawa 923-1292, Japan.
a) hirotaka@econ.kyushu-u.ac.jp
b) otachi@jaist.ac.jp

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any $W \geq 1$. This ratio is almost tight, because it was shown that, for sufficiently large values of W, Min W-Light is NP-hard to approximate within $\ln(W+1) - O(\log\log W)$.

- For sufficiently large values of W, Max W-Heavy is NP-hard to approximate within $(n/W)^{1/2-\varepsilon}$ for any $\varepsilon > 0$. Note that the best known polynomial-time approximation ratio for Max W-Heavy is $W+1$ [3].

Due to the work, the general (in)approximability of the problems is almost clear. In this paper, we thus investigate the problem from another aspect, that is, graph classes.

We first show that both Max W-Light and Max W-Heavy can be solved in linear time for graphs of bounded treewidth. We use the optimization version [1] of Courcelle's theorem [12] for Max W-Light, and a standard dynamic programming approach for Max W-Heavy. We then investigate graphs of unbounded treewidth. As we have seen, if we do not consider any restriction, the problem becomes intractable. Here, we focus on graph classes with some nice properties.

Fomin, Todinca, and Villanger [14] present a metatheorem that can be seen as an extension of Courcelle's theorem. It provides a polynomial-time algorithm for finding a maximum induced subgraph of bounded treewidth satisfying a counting monadic second order logic formula from a given graph with polynomially many potential maximal cliques. In their paper, they list known graph classes with polynomially many potential maximal cliques, e.g., weakly chordal graphs, polygon-circle graphs, circular-arc graphs, and d-trapezoid graphs.

Unfortunately, we failed to apply the metatheorem of Fomin et al. to our problems, but succeed in showing that it can be applied to Max W-Light for some known graph classes with polynomially many potential maximal cliques. As a byproduct, we show that Max W-Heavy can be solved in linear time for the same graph classes.

1.1 Related work

Graph orientations that optimize certain objective functions involving the resulting directed graph or that satisfy some special property such as acyclicity [30] or k-edge connectivity [11], [27], [28] have many applications to graph theory, combinatorial optimization, scheduling (load balancing), resource allocation, and efficient data structures. For example, an orientation that minimizes the maximum outdegree [5], [10], [31] can be used to support fast vertex adjacency queries in a sparse graph by storing each edge in exactly one of its two incident vertices' adjacency lists while ensuring that all adjacency lists are short [10]. There are many optimization criteria for graph orientation other than these. See [2] or Chapter 61 in [29] for more details and additional references.

On the other hand, degree-constrained graph orientations [15], [16], [18], [25] arise when a lower degree bound $W^l(v)$ and an upper degree bound $W^u(v)$ for each vertex v in the graph are specified in advance or as part of the input, and the outdegree of v in any valid graph orientation is required to lie in the interval $[W^l(v), \ldots, W^u(v)]$. Obviously, a graph does not always have such an orientation, and in this case, one might want to compute an orientation that best fits the outdegree constraints according to some well-defined criteria [2], [3]. In case $W^l(v) = 0$ and $W^u(v) = W$ for every vertex v in the input graph, where W is a non-negative integer, and the objective is to maximize (resp., minimize) the number of vertices that satisfy (resp., violate) the outdegree constraints, then we obtain Max W-Light (resp., Min $(W+1)$-Heavy). Similarly, if $W^l(v) = W$ and $W^u(v) = \infty$ for every vertex v in the input graph, then we obtain Max W-Heavy and Min $(W-1)$-Light.

2. Preliminaries

We say two vertices $u, v \in V(G)$ are *adjacent in* G if $\{u, v\} \in E(G)$. Let $N_G(u)$ be the set of all vertices that are adjacent to u in G, i.e., $N_G(u) = \{v : \{u, v\} \in E\}$. The *degree of* u *in* G is denoted by $d_G(u)$. We define $\delta(G) = \min\{d_G(u) : u \in V(G)\}$. The *degeneracy* of a graph G, denoted by $\hat{\delta}(G)$, is the maximum of the minimum degrees over all subgraphs of G. That is, $\hat{\delta}(G) = \max_{H \subseteq G} \delta(H)$. For any $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$. If $G[U]$ is a complete graph, then U is a *clique* of G. The size of a maximum clique in G is denoted by $\omega(G)$. Let $\omega_{\mathsf{bi}}(G)$ the maximum integer k such that G has a subgraph isomorphic to the complete bipartite graph $K_{k,k}$. From the definition, $\omega(G) - 1$ and $\omega_{\mathsf{bi}}(G)$ are lower bounds of $\hat{\delta}(G)$.

2.1 Minimal triangulations and potential maximal cliques

A graph is *chordal* (or *triangulated*) if it has no induced cycle of length 4 or more. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $G' = (V, E')$ such that $E \subseteq E'$. A triangulation G' of G is *minimal* if no proper subgraph of G' is a triangulation of G. The *treewidth* of G, denoted by $\mathbf{tw}(G)$, is the minimum integer t such that there is a (minimal) triangulation H of G with the maximum clique size $t+1$. A vertex set $P \subseteq V(G)$ is a *potential maximal clique* of G if P is a maximal clique in some minimal triangulation of G. The set of all potential maximal cliques of G is denoted by Π_G. A vertex set $S \subseteq V(G)$ is an a, b*-separator* for $a, b \in V(G)$ if a and b are in different components in $G - S$. An a, b-separator is *minimal* if no proper subset of it is an a, b-separator. A vertex set is a *minimal separator* if it is a minimal a, b-separator for some pair a, b. The set of all minimal separators of G is denoted by Δ_G. By the following proposition, if graphs from some class have a polynomial number of minimal separators, these graphs also have a polynomial number of potential maximal cliques.

Proposition 2.1 (Bouchitté and Todinca [9]). *For every n-vertex graph G, it holds that* $|\Pi_G| \leq n|\Delta_G|^2 + n|\Delta_G| + 1$.

2.2 W-light/heavy orientability

For an integer $W \geq 0$, an orientation of a graph is called a W-*light orientation* if the maximum outdegree is at most W. If a W-light orientation exists, we say that the graph is W-*light orientable*. By replacing "at most" with "at least" in these definitions, we similarly define W-*heavy orientations* and W-*heavy orientable* graphs.

It can be seen that the problem of finding a maximum W-light orientable induced subgraph is polynomially equivalent to Max W-Light.

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Lemma 2.2. *If a maximum W-light orientable induced subgraph of G can be found in $O(f(m,n))$ time, then* Max W-Light *can be solved in $O(f(m,n) + m^{1.5})$ time, where m and n are the number of edges and vertices in G, respectively.*

Proof. First we find a maximum W-light orientable induced subgraph H of G in $O(f(m,n))$ time. Next we compute a W-light orientation Λ' of H in $O(m^{1.5})$ time [6]. Finally, to obtain an orientation Λ of G, we extend Λ' in linear time as follows: orient each edge $\{u,v\} \in E(G)$ as (u,v) if $u \in V(G) \setminus V(H)$ and $v \in V(H)$; orient arbitrarily the edges completely in $V(G) \setminus V(H)$.

We now show that Λ is an optimal solution of Max W-Light. The extension does not increase the outdegree of any vertex in $V(H)$. Thus the number of W-light vertices in $\Lambda(G)$ is at least $|V(H)|$. Suppose to the contrary that there is an orientation Λ_{OPT} with the W-light vertices F such that $|F| > |V(H)|$. Now $G[F]$ is a W-light orientable graph because Λ_{OPT} restricted to $G[F]$ is a W-light orientation. This contradicts the optimality of H. □

3. Metatheorems

In this section we present meththeorems for Max W-Light and Max W-Heavy. We apply them to some well-studied graph classes in the next section.

3.1 Max W-Light

We first introduce the *monadic second order logic* (MSO) of graphs. The syntax of MSO of graphs includes (i) the logical connectives $\vee$, $\wedge$, $\neg$, $\Leftrightarrow$, $\Rightarrow$, (ii) variables for vertices, edges, vertex sets, and edge sets, (iii) the quantifiers $\forall$ and $\exists$ applicable to these variables, and (iv) the following binary relations:

- $u \in U$ for a vertex variable u and a vertex set variable U;
- $d \in D$ for an edge variable d and an edge set variable D;
- $\mathbf{inc}(d,u)$ for an edge variable d and a vertex variable u, where the interpretation is that d is incident with u;
- equality of variables.

In the *counting monadic second order logic* (CMSO) of graphs, we have an additional sentence of checking the cardinality of a set modulo some constant. We also have MSO of digraphs in which we have $\mathbf{arc}(u,a,v)$ instead of $\mathbf{inc}(d,u)$, where $\mathbf{arc}(u,a,v)$ implies that a is an arc from u to v.

For a property $\mathcal{P}$ of digraphs, let $\mathbf{und}(\mathcal{P})$ be the property of graphs such that a graph G satisfies $\mathbf{und}(\mathcal{P})$ if and only if an orientation of G satisfies $\mathcal{P}$. Courcelle [13] showed that if $\mathcal{P}$ is an MSO-expressible property of finite directed graphs, then $\mathbf{und}(\mathcal{P})$ is MSO-expressible. It is easy to see that for a fixed W, we can express in MSO the property of digraphs having maximum outdegree at most W. Thus we have the following lemma.

Lemma 3.1. *For a fixed W, the property of being W-light orientable can be expressed in MSO.*

It is known that for a fixed MSO formula on a graph and its vertex subsets, one can find in linear time a maximum vertex subset satisfying the formula if the input graph belongs to a graph class of bounded treewidth (see [1], [12]). This fact and Lemma 3.1 imply the following corollary.

Corollary 3.2. *For a fixed W,* Max W-Light *can be solved in linear time for a graphs of bounded treewidth.*

Recently, Fomin, Todinca, and Villanger [14] have extended the optimization version of Courcelle's metatheorem as follows.

Proposition 3.3 (Fomin, Todinca, and Villanger [14])**.** *For any fixed t and a CMSO-expressible property $\mathcal{P}$, the following problem can be solved in polynomial time for classes of graphs with a polynomial number of potential maximal cliques: Given a graph G, find a maximum induced subgraph H of treewidth at most t that has the property $\mathcal{P}$.*

The new metatheorem above is quite powerful and allows to solve many problems for graphs with polynomially many potential maximal cliques. (See [14] for applications.) However, we cannot apply it to our problem Max W-Light in general because W-light orientable graphs may have large treewidth. For example, grid graphs are 2-light orientable but have unbounded treewidth.

In the following, we show that with an additional restriction to graph classes, we can apply the metatheorem of Fomin, Todinca, and Villanger to Max W-Light.

Lemma 3.4. *Every W-light orientable graph has degeneracy at most $2W$.*

Proof. We prove the contrapositive. Let G be a graph with $\hat{\delta}(G) > 2W$. There is a subgraph H of G such that $\delta(H) > 2W$. Since the average degree $2|E(H)|/|V(H)|$ of H is at least $\delta(H)$, we have $|E(H)|/|V(H)| > W$. Thus for any orientation Λ of G,

$$\max_{u \in V(G)} d^+_\Lambda(u) \geq \max_{u \in V(H)} d^+_\Lambda(u) \geq \sum_{u \in V(H)} d^+_\Lambda(u)/|V(H)| \geq |E(H)|/|V(H)| > W.$$

This implies that G is not W-light orientable. □

Theorem 3.5. *For a fixed W,* Max W-Light *can be solved in polynomial time for a graph class C with a polynomial number of potential maximal cliques if the treewidth of each graph in C is bounded from above by a function of its degeneracy.*

Proof. Let f be a function such that $\mathbf{tw}(G) \leq f(\hat{\delta}(G))$ for each $G \in C$. By Lemma 3.4, a W-light orientable graph in C has treewidth at most $f(2W)$. Thus a maximum W-light orientable induced subgraph a graph in C can be found in polynomial time by Proposition 3.3 and Lemma 3.1. By Lemma 2.2, we can solve Max W-Light in polynomial time. □

3.2 Max W-Heavy

Unlike Max W-Light, the problem Max W-Heavy does not seem to be equivalent to the problem of finding a maximum orientable induced subgraph. We here present a way of directly finding an orientation with as many W-heavy vertices as possible for a graph in a graph class with a nice property.

A *tree decomposition* of a graph $G = (V,E)$ is a pair $(\{X_i : i \in I\}, T = (I,F))$ such that each X_i, called a *bag*, is a subset of V, and T is a tree such that

- for each $v \in v$, there is $i \in I$ with $v \in X_i$;
- for each $\{u,v\} \in E$, there is $i \in I$ with $u,v \in X_i$;
- for $i,j,k \in I$, if j is on the i,k-path in T, then $X_i \cap X_k \subseteq X_j$.

The *width* of a tree decomposition is the size of a maximum bag minus one. A graph has treewidth at most t if and only if it has a tree decomposition of width at most t. A tree decomposition

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$(\{X_i : i \in I\}, T = (I, F))$ is *nice* if

- T is rooted at a node $r \in I$;
- every node $i \in I$ has at most two children;
- if a node i has two children j, k, then $X_i = X_j = X_k$ (such a node is called a *join node*);
- if a node i has one children j, then either
  - $X_i = X_j \cup \{v\}$ for some $v \in V$ (i is called a *introduce node*), or
  - $X_i = X_j \setminus \{v\}$ for some $v \in V$ (i is called a *forget node*);
- if a node i is a leaf, then $X_i = \{v\}$ for some $v \in V$.

Bodlaender [7] shows that given a graph of treewidth t, a tree decomposition of width t can be computed in linear time Also it is known that given a tree decomposition of a graph G of width t it is possible to transform it in linear time into a nice tree decomposition of G of width t and with at most $4n$ nodes [20].

Lemma 3.6. *For a fixed $W \geq 1$,* Max W-Heavy *can be solved in linear time for graphs of bounded treewidth.*

Proof. We show that the maximum number of W-heavy vertices over all orientations can be computed in linear time. It is straightforward to modify the dynamic program below for computing an actual orientation in the same running time.

Let $G = (V, E)$ be a graph of treewidth t. We first compute a nice tree decomposition $(\{X_i : i \in I\}, T = (I, F))$ with the root r in linear time. For each $i \in I$, let $V_i = X_i \cup \bigcup_j X_j$, where j runs through all descendants of i in T. Let G_i denote the graph $G[V_i]$, and let B_i denote the graph $G[X_i]$.

For each node i, orientation λ of B_i, and mapping $\phi\colon X_i \to \{0, \ldots, W\}$, we define the value $\mathsf{heavy}(i, \lambda, \phi)$ as the maximum number of W-heavy vertices in G_i with an orientation Λ such that Λ agrees with λ in B_i and each vertex $v \in X_i$ has at least $\phi(v)$ out-neighbors in $V_i - X_i$ under Λ. If no such orientation Λ exists, then we set $\mathsf{heavy}(i, \lambda, \phi) = -\infty$. There are at most $|I| \cdot 2^{\binom{t+1}{2}} \cdot (W + 1)^{t+1} \in O(n)$ entries $\mathsf{heavy}(i, \lambda, \phi)$, and $\max_{\lambda,\phi} \mathsf{heavy}(r, \lambda, \phi)$ is the desired value. Therefore, to prove the lemma, it suffices to show that each entry can be computed in time depends only on W and t.

We compute all entries $\mathsf{heavy}(i, \lambda, \phi)$ by a bottom-up dynamic program. For a leaf node i, $\mathsf{heavy}(i, \lambda, \phi) = 0$. In what follows, we assume that the values for the children of the node currently considered are already computed.

Forget nodes: Let i be a forget node with child j and $X_i = X_j \setminus \{u\}$. Then

$$\mathsf{heavy}(i, \lambda, \phi) = \max_{\mu,\psi} \mathsf{heavy}(j, \mu, \psi),$$

where the maximum is taken over all orientations μ of B_j and all mappings $\psi\colon X_j \to \{0, \ldots, W\}$ such that μ agrees with λ in B_i, $\phi(v) = \psi(v) + 1$ for each $v \in X_j$ with $(u, v) \in \mu(B_j)$, and $\phi(v) = \psi(v)$ otherwise for the other vertices $v \in X_j$. There are at most 2^t candidates of pairs (μ, ψ) consistent with (λ, ϕ) and consistency of each pair can be checked in $O(t)$ time. Thus the value $\mathsf{heavy}(i, \lambda, \phi)$ for a forget node i can be computed in $O(2^t \cdot t)$ time.

Introduce nodes: Let i be an introduce node with child j and $X_i = X_j \cup \{u\}$. From the definition of tree decompositions, u has neighbors only in X_j in the graph G_i. That is, $N_{G_i}(u) \subseteq X_j$. Hence, $\mathsf{heavy}(i, \lambda, \phi) = -\infty$ if $\phi(u) \neq 0$. If $\phi(u) = 0$, we can compute the value as follows:

$$\mathsf{heavy}(i, \lambda, \phi) = \mathsf{heavy}(j, \lambda', \phi') + |S_{i,\lambda,\phi}|,$$

where λ' is the orientation of B_j obtained by restricting λ to B_j, $\phi'(v) = \phi(v)$ for $v \in B_j$, $S_{i,\lambda,\phi} = \{v \in X_i : ld^+_\lambda(v) + \phi(v) \geq W\} \setminus \{v \in X_j : ld^+_{\lambda'}(v) + \phi'(v) \geq W\}$. The vertices in $S_{i,\lambda,\phi}$ contribute to $\mathsf{heavy}(i, \lambda, \phi)$ but not to $\mathsf{heavy}(j, \lambda', \phi')$, and the other vertices in G_i contribute to both $\mathsf{heavy}(i, \lambda, \phi)$ and $\mathsf{heavy}(j, \lambda', \phi')$. Thus the formula correctly compute the value $\mathsf{heavy}(i, \lambda, \phi)$. Clearly, it can be computed in $O(t)$ time.

Join nodes: Let i be a join node with children j and k. It holds that

$$\mathsf{heavy}(i, \lambda, \phi) = \max_{\psi,\psi'} \left(\mathsf{heavy}(j, \lambda, \psi) + \mathsf{heavy}(k, \lambda, \psi') + |R_{i,\lambda,\psi,\psi'}|\right),$$

where the maximum is taken over all mappings $\psi, \psi'\colon X_j \to \{0, \ldots, W\}$ such that $\phi(v) = \psi(v) + \psi'(v)$ for each $v \in X_i$, and the set $R_{i,\lambda,\psi,\psi'}$ is defined as $R_{i,\lambda,\psi,\psi'} = \{v \in X_i : ld^+_\lambda(v) + \psi(v) + \psi'(v) \geq W\} \setminus \{v \in X_i : ld^+_\lambda(v) + \max\{\psi(v), \psi'(v)\} < W\}$. The vertices in $S_{i,\lambda,\psi,\psi'}$ contribute only to $\mathsf{heavy}(i, \lambda, \phi)$, and the other vertices in G_i contribute to all $\mathsf{heavy}(i, \lambda, \phi)$, $\mathsf{heavy}(j, \lambda, \psi)$, and $\mathsf{heavy}(j, \lambda, \psi')$. Thus the formula is correct. It can be computed in time $O((W+1)^{t+1})$ as $\psi(v) \in \{0, \ldots, W\}$ for each vertex $v \in X_i$. □

Proposition 3.7 ([4])**.** *A W-heavy orientation of a graph of minimum degree at least $2W + 1$ can be found in linear time.*

Theorem 3.8. *For a fixed W,* Max W-Heavy *can be solved in linear time for a graph class C if the treewidth of each graph in C is bounded from above by a function of its degeneracy.*

Proof. Let f be a function such that $\mathbf{tw}(G) \leq f(\hat{\delta}(G))$ for each $G \in C$. Let $G \in C$ be a graph with n vertices. Let $(v_1, v_2, \ldots, v_n)$ be an ordering of $V(G)$ such that for each i, the vertex v_i has the minimum degree in G_i, where $G_i = G[\{v_j : i \leq j \leq n\}]$. Let h be the first index such that $d_{G_h}(v_h) \geq 2W + 1$. If there is no such index, we set $h = n + 1$.

Let $H_< = G[\{v_j : 1 \leq j < h\}]$. We obtain $H'_<$ from $H_<$ by adding some vertices and edges as follows: For each vertex v in $H_<$, we add a clique C_v of size $2W + 1$; Add edges from v to arbitrarily chosen $d_G(v) - d_{H_<}(v)$ vertices in C_v.

Claim 3.9. $\mathbf{tw}(H'_<) \leq \max\{f(\hat{\delta}(G)), 2W + 1\}$.

Proof of Claim. Omitted. □

The claims above and Lemma 3.6 together imply that an orientation Λ of $H'_<$ with the maximum number of W-heavy vertices can be found in linear time. Also, by Proposition 3.7, a W-heavy orientation Λ' of G_h can be found in linear time. Now, from Λ and Λ', we can construct an orientation of G with the maximum number of W-heavy vertices can be found in linear time. The detail is omitted. □

4. Graph classes

In this section, we show that Theorems 3.5 and 3.8 can be applied to some graph classes with polynomial many potential maximal cliques. More precisely, we show the following theorem.

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Theorem 4.1. *For a fixed W,* Max W-Light *can be solved in polynomial time for the classes of chordal graphs, d-trapezoid graphs, circular-arc graphs, and chordal bipartite graphs.*

Theorem 4.2. *For a fixed W,* Max W-Heavy *can be solved in linear time for the classes of chordal graphs, d-trapezoid graphs, circular-arc graphs, and chordal bipartite graphs.*

To prove Theorems 4.1 and 4.2, we show for each graph class that it satisfies conditions of Theorems 3.5 and 3.8 in the following subsections.

4.1 Chordal graphs

It is well known that a chordal graph of n vertices has at most n maximal cliques (see [19]). Since a chordal graph is the unique minimal triangulation of itself, the number of potential maximal cliques is at most n for every n-vertex chordal graph. From the definition of chordal graphs, the following equality follows.

Proposition 4.3 (Folklore)**.** *For every chordal graph G,* $\mathbf{tw}(G) = \hat{\delta}(G) = \omega(G) - 1$.

4.2 d-trapezoid graphs

The *co-comparability graph* of a partial order $(V, \prec)$ is a graph with the vertex set V in which two vertices u and v are adjacent if and only if they are incomparable, that is, $u \not\prec v$ and $v \not\prec u$. A partial order $(V, \prec)$ is an *interval order* if each element $v \in V$ can be represented by an interval $[l_v, r_v]$ such that $u \prec v$ if and only if $r_u < l_v$. A graph is a *d-trapezoid graph* if it is the co-comparability graph of a partial order defined as the intersection of d interval orders [8].

It is known that for a constant d, every d-trapezoid graph of n vertices has at most $(2n-3)^{d-1}$ minimal separators [24].

Habib and Möhring showed in the proof of Theorem 3.4 in [17] that for every d-trapezoid graph G, $\mathbf{tw}(G) \le 4d \cdot \omega_{\mathsf{bi}}(G) - 1$. This gives the following fact as a direct corollary.

Proposition 4.4 ([17])**.** *For every d-trapezoid graph G,* $\mathbf{tw}(G) \le 4d \cdot \hat{\delta}(G) - 1$.

4.3 Circular-arc graphs

A graph is an *interval graph* if it is the intersection graph of interval on a line. Every interval graph is a chordal graph [26]. A graph is a *circular-arc graph* if it is the intersection graph of arcs on a circle. Every circular-arc graph has at most $2n^2 - 3n$ minimal separators [22].

Lemma 4.5. *For every circular-arc graph G,* $\mathbf{tw}(G) \le 2\omega(G) - 1$.

Proof. Let p be a point on the circle in a circular-arc representation of G, and S_p be the vertices that correspond to the arcs containing p in the representation. The set S_p is a clique, and thus $|S_p| \le \omega(G)$. Let $G' = G - S_p$. Since G' has a circular-arc representation in which the arcs do not cover the entire circle (especially they do not cover the point p), G' is an interval graph. By Proposition 4.3, $\mathbf{tw}(G') = \omega(G') - 1 \le \omega(G) - 1$. Since a removal of a vertex can decrease the treewidth by at most 1, we can conclude that $\mathbf{tw}(G) \le \mathbf{tw}(G') + |S_p| \le 2\omega(G) - 1$. □

4.4 Chordal bipartite graphs

A bipartite graph is a *chordal bipartite graph* if it has no induced cycle length 6 or more. Every chordal bipartite graph has $O(m+n)$ minimal separators [23]. We show in this subsection that for every chordal bipartite graph G, $\mathbf{tw}(G) \le 2\hat{\delta}(G) - 1$.

Let $G = (X, Y; E)$ be a chordal bipartite graph. We call (A, B) with $A \subseteq X$ and $B \subseteq Y$ a *biclique* if $G[A \cup B]$ is a complete bipartite graph. A biclique (A, B) is *maximal* if there is no other biclique (A', B') satisfying $A \cup B \subsetneq A' \cup B'$. Let $\mathcal{M}_{\mathsf{bi}}(G)$ be the set of maximal bicliques (A, B) of G with $\min\{|A|, |B|\} \ge 2$.

Lemma 4.6. *If (A, B) and (A', B') are distinct elements of $\mathcal{M}_{\mathsf{bi}}(G)$, then $A \ne A'$ and $B \ne B'$.*

Proof. Suppose that $A = A'$. From the assumption, it follows that $(A, B) \ne (A, B')$ and $(A, B), (A, B') \in \mathcal{M}_{\mathsf{bi}}(G)$. This implies that $B' \not\subseteq B$, and thus $B \subsetneq B \cup B'$. Hence $A \cup B \subsetneq A \cup (B \cup B')$. However, $(A, B \cup B') \in \mathcal{M}_{\mathsf{bi}}(G)$ holds because $A = A'$. This contradicts the maximality of (A, B). The other case $B = B'$ can be ruled out in the same way. □

Two maximal bicliques (A_1, B_1) and (A_2, B_2) *cross* if either $A_1 \supsetneq A_2$ and $B_1 \subsetneq B_2$, or $A_1 \subsetneq A_2$ and $B_1 \supsetneq B_2$. If $c\colon \mathcal{M}_{\mathsf{bi}}(G) \to 2^{V(G)}$ is a mapping such that $c(A, B) \in \{A, B\}$ for each $(A, B) \in \mathcal{M}_{\mathsf{bi}}(G)$, then we call the family $\mathcal{C} = \{c(A, B) : (A, B) \in \mathcal{M}_{\mathsf{bi}}(G)\}$ a *biclique coloring* of G. A biclique coloring $\mathcal{C}$ is *feasible* if for each pair $(A_1, B_1), (A_2, B_2) \in \mathcal{M}_{\mathsf{bi}}(G)$ that cross with $A_1 \supsetneq A_2$ and $B_1 \subsetneq B_2$, not both A_1 and B_2 are in $\mathcal{C}$.

For a family of vertex sets $\mathcal{S} \subseteq 2^V$, we denote by $G_{\mathcal{S}}$ the graph obtained by making each $S \in \mathcal{S}$ a clique.

Proposition 4.7 (Kloks and Kratsch [21])**.** *If $\mathcal{C}$ is a feasible biclique coloring of a chordal bipartite graph G, then $G_{\mathcal{C}}$ is chordal.*

Let $\mathcal{C}_{\min}$ be a family that contains a smaller one of A and B for each $(A, B) \in \mathcal{M}_{\mathsf{bi}}(G)$. That is, $\mathcal{C}_{\min}(G) = \{\arg\min_{C \in \{A,B\}} |C| : (A, B) \in \mathcal{M}_{\mathsf{bi}}(G)\}$, where the ties in arg min are broken arbitrarily. Note that for each $(A, B) \in \mathcal{M}_{\mathsf{bi}}(G)$, $A \in \mathcal{C}_{\min}(G)$ implies $|A| \le |B|$ by Lemma 4.6.

Lemma 4.8. *$\mathcal{C}_{\min}(G)$ is a feasible biclique coloring for every chordal bipartite graph G.*

Proof. Suppose to the contrary that

(1) $(A_1, B_1), (A_2, B_2) \in \mathcal{M}_{\mathsf{bi}}(G)$ cross with $A_1 \supsetneq A_2$ and $B_1 \subsetneq B_2$, and

(2) $A_1, B_2 \in \mathcal{C}_{\min}(G)$.

Now (1) gives $|A_1| > |A_2|$ and $|B_1| < |B_2|$, and (2) gives $|A_1| \le |B_1|$ and $|B_2| \le |A_2|$. They cause a contradiction $|A_1| \le |B_1| < |B_2| \le |A_2| < |A_1|$. □

Proposition 4.9 (Kloks and Kratsch [21])**.** *Let $\mathcal{C}$ be a feasible biclique coloring of a chordal bipartite graph $G = (X, Y; E)$. Let K be a maximal clique in $G_{\mathcal{C}}$ with $|K| > 2$. Let $K_X = K \cap X$ and $K_Y = K \cap Y$. If $|K_X| \ge 2$, then one of the following two cases holds:*

(1) $|K_Y| = 1$ and there exists $(A, B) \in \mathcal{M}_{\mathsf{bi}}(G)$ such that $K_X = A$ and $A \in \mathcal{C}$.

(2) $|K_Y| > 1$ and there exist $(A_1, B_1), (A_2, B_2) \in \mathcal{M}_{\mathsf{bi}}(G)$ with $A_1, B_2 \in \mathcal{C}$ such that $K_X \subseteq A_1$ and $K_Y \subseteq B_2$.

Lemma 4.10. *Let $G = (X, Y; E)$ be a chordal bipartite graph and $\mathcal{C} = \mathcal{C}_{\min}(G)$. Then $\omega(G_{\mathcal{C}}) \le 2 \cdot \omega_{\mathsf{bi}}(G)$.*

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Proof. By Lemma 4.8, C is a feasible biclique coloring of G. Let K be a maximal clique in G_C with $|K| > 2$. Let $K_X = K \cap X$ and $K_Y = K \cap Y$. Assume without loss of generality that $|K_X| \geq 2$. Now by Proposition 4.9, one of the following two cases holds:

(1) $|K_Y| = 1$ and there exists $(A, B) \in \mathcal{M}_{\mathsf{bi}}(G)$ such that $K_X = A$ and $A \in C$.

(2) $|K_Y| > 1$ and there exist $(A_1, B_1), (A_2, B_2) \in \mathcal{M}_{\mathsf{bi}}(G)$ with $A_1, B_2 \in C$ such that $K_X \subseteq A_1$ and $K_Y \subseteq B_2$.

In the first case, $|A| \leq |B|$ holds, and thus $|K| = |A|+1 \leq \omega_{\mathsf{bi}}(G)+1$. In the second case, $|A_1| \leq |B_1|$ and $|B_2| \leq |A_2|$ together imply that $|K| \leq |A_1| + |B_2| \leq 2 \cdot \omega_{\mathsf{bi}}(G)$. □

By Proposition 4.7, G_C is chordal if $C = C_{\min}(G)$. By Proposition 4.3 and the lemma above, the following corollary follows.

Corollary 4.11. *For every chordal bipartite graph G,* $\mathbf{tw}(G) \leq 2 \cdot \omega_{\mathsf{bi}}(G) - 1$.

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