Regular Paper

On the Global Convergence of Some Iterative Formulas

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We showed two types of third order iterative formulas containing two parameters. Let $f(x)$ be a polynomial with only real zeros or an entire function of a certain type with only real zeros. Then we established that the one type of the above-mentioned iterative formulas converges globally and monotonically to the zeros of the $f(x)$. The purpose of this paper is to show that the other type also converges globally and monotonically to the zeros of the $f(x)$.

1. Introduction

Ostrowski[1)] and Traub[2)] have shown various types of iterative formulas for the computation of the numerical solutions of the nonlinear scalar equation

$$f(x)=0 \tag{1.1}$$

where $f(x)$ is a real function of the real variable x. In Ref. 3)～6), we also have shown various types of iterative formulas.

Let $f(x)$ be a polynomial of exact degree $r>1$ with only real zeros given by the following form:

$$f(x)=\prod_{k=1}^{r}(x-\alpha_k). \tag{1.2}$$

Then, it has been shown that Ostrowski's method (Ref. 1), pp. 110-115), Laguerre's method (Ref. 1), pp. 353-362), and Hansen and Patrick's methods[7)] converge globally and monotonically to the zeros of $f(x)$.

Next, let $f(x)$ be given by the following form:

$$f(x)=x^p \exp(a+bx-cx^2) \cdot\prod_k\left(1-\frac{x}{\alpha_k}\right)e^{x/\alpha_k} \tag{1.3}$$

where p is a non-negative integer, a, b, c are real with $c\geqq 0$, and α_k are real. If the number of the α_k is infinite, then $\Sigma\alpha_k^{-2}<\infty$. Furthermore, if the number of the α_k is finite, then we require that there be at least one α_k for $p\geqq 1$, and at least two for $p=0$.

Then, it has been shown that Ostrowski's method (Ref. 1), pp. 124-126), Hally's method[8)], and Hansen and Patrick's methods[7)] converge globally and monotonically to the zeros of $f(x)$.

In Ref. 6), we considered the following type of iterative formulas:

$$x_{n+1}=\phi(x_n),\ n=0, 1, \cdots \tag{1.4}$$

where $\phi(x)=x-hR(X)$, $h\equiv h(x)=\frac{f(x)}{f'(x)}$, $X\equiv X(x)=h\frac{f''(x)}{f'(x)}$,

and $R(t)$ is a function of t. Then, it was shown that the order of convergence for Eq. (1.4) is equal to 3 for all simple zeros of Eq. (1.1), iff $R(0)=1$ and $R'(0)=1/2$.

Furthermore, we gave two examples of $R(X)$ in the form:

Ex. 1 $R(X)=\dfrac{(\theta+\frac{1}{2})X+1}{\beta X^2+\theta X+1}$ (β, θ: parameters),

Ex. 2 $R(X)=\dfrac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}$ (a, b: parameters).

In Ref. 6), we established that iterative formulas for $R(X)$ in Ex. 1 converge globally and monotonically to the zeros of $f(x)$ given by both the form (1.2) and the form (1.3). In 2 and 3, we will consider the monotonic global convergence of iterative formulas for $R(X)$ of Ex. 2.

2. Monotonicity of Convergence

We will consider the iterative methods for $R(X)$ of Ex. 2 that form the sequence x_n by the iterative rule:

$$x_{n+1}=x_n-K(x_n),\ n=0, 1, \cdots \tag{2.1}$$

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where

$$K(x)=h\frac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}.$$

At first, let $f(x)$ be given by the form (1.3). Then, taking the logarithmic derivative of (1.3) and differentiating it, we have

$$\frac{f'(x)}{f(x)}=\frac{p}{x}+b-2cx+\sum_k\left(\frac{1}{x-\alpha_k}+\frac{1}{\alpha_k}\right), \tag{2.2}$$

$$-\left(\frac{f'(x)}{f(x)}\right)'=\frac{\{f'(x)\}^2-f(x)f''(x)}{\{f(x)\}^2}=\frac{p}{x^2}+2c+\sum_k\frac{1}{(x-\alpha_k)^2}. \tag{2.3}$$

Let the distinct zeros of $f(x)$ be ordered consecutively so that $\zeta_0<\zeta_1$. Then, since the right hand side of (2.3) is positive, it follows that $\frac{f'(x)}{f(x)}$ is monotonically decreasing in the open interval (ζ_0, ζ_1).

Furthermore, since $\lim_{x\to\zeta_0+0}\frac{f'(x)}{f(x)}=+\infty$, and $\lim_{x\to\zeta_1-0}\frac{f'(x)}{f(x)}=-\infty$, it follows that $f'(x)$ has exactly one zero ζ_0' such that $\zeta_0<\zeta_0'<\zeta_1$. Then, for $\forall x\in(\zeta_0, \zeta_0')\cup(\zeta_0', \zeta_1)$, we can define the associated zero $\alpha(x)$ of $f(x)$ to be $\alpha(x)=\zeta_0$ if $\zeta_0<x<\zeta_0'$ and $\alpha(x)=\zeta_1$ if $\zeta_0'<x<\zeta_1$.

Furthermore $\frac{f'(x)}{f(x)}$ and $x-\alpha(x)$ have the same sign. It now follows from (2.3) that

$$1-X=h^2\left\{\frac{p}{x^2}+2c+\sum_k\frac{1}{(x-\alpha_k)^2}\right\}>0\ (\zeta_0<x<\zeta_1). \tag{2.4}$$

It follows from (2.4) that for any real x such that $f(x)f'(x)\neq 0$, we have

$$1-X-\frac{h^2}{\{x-\alpha(x)\}^2}=h^2\left[\frac{p}{x^2}+2c+\sum_k\frac{1}{(x-\alpha_k)^2}-\frac{1}{\{x-\alpha(x)\}^2}\right]>0. \tag{2.5}$$

Hence, it follows from (2.5) that

$$|x-\alpha(x)|>\frac{|h|}{\sqrt{1-X}}. \tag{2.6}$$

We will need later the following lemma:

Lemma 1. If $-\sqrt{b}<a\leqq 0$, then we have

$$0<\frac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}\leqq\frac{1}{\sqrt{1-X}}\quad(X<1). \tag{2.7}$$

Proof. From the assumption, we have $a+\sqrt{b}>0$.

It then follows that for $X<1$,

$$b-\sqrt{b}(a+\sqrt{b})X>b-\sqrt{b}(a+\sqrt{b})=-a\sqrt{b}\geqq 0.$$

Putting $\Phi(X)=a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}$, we have $\Phi(X)>\Phi(1)\ (X<1)$.

On the other hand, since $\Phi(1)=a+\sqrt{-a\sqrt{b}}>a+|a|=0$,

$$\Phi(X)>0\quad(X<1). \tag{2.8}$$

In order to prove that we have (2.7), it suffices to prove that the following inequality holds:

$$(a+\sqrt{b})\sqrt{1-X}\leqq\Phi(X),\quad(X<1) \tag{2.9}$$

In order to prove that (2.9) holds, we put

$$g(X)=\Phi(X)-(a+\sqrt{b})\sqrt{1-X}.$$

Then,

$$g'(X)=\frac{-\sqrt{b}(a+\sqrt{b})}{2\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}+\frac{a+\sqrt{b}}{2\sqrt{1-X}}=\frac{1}{2}(a+\sqrt{b})\left(\frac{1}{\sqrt{1-X}}-\frac{1}{\sqrt{1-X-\frac{a}{\sqrt{b}}X}}\right).$$

Since $\frac{a}{\sqrt{b}}\leqq 0$, we have $g'(X)\leqq 0$ if $X<0$ and $g'(X)\geqq 0$ if $X\geqq 0$. In addition, since $g(0)=0$, we have $g(X)\geqq 0$.

Hence (2.9) is proved. Consequently, from (2.8) and (2.9), we have (2.7).

Thus Lemma 1 is completely proved.

It now follows from (2.6) and Lemma 1 that

$$|x-\alpha(x)|>|K(x)| \tag{2.10}$$

Then, on the monotonic convergence of the rule (2.1), we have:

Theorem 1. Let $f(x)$ be given by the form (1.3). Then, if $-\sqrt{b}<a\leqq 0$ and if we take the real starting value in (2.1) x_0 such that $f'(x_0)f(x_0)\neq 0$, and x_0 is neither less nor greater than all zeros of $f(x)$, the sequence x_n in (2.1) converge monotonically to $\alpha(x_0)$.

Proof. Assume that we take the starting value x_0 such that $\zeta_0'<x_0<\zeta_1$.

Then, we have $\alpha(x_0)=\zeta_1, f(x_0)/f'(x_0)<0$, and $K(x_0)<0$. Next, applying (2.1) and (2.10), we have

$$x_0<x_1<\alpha(x_0),$$

and by the definition of $\alpha(x)$, $\alpha(x_0)\equiv\alpha(x_1)$.

By repetition of the same argument, we have

$$x_0 < x_1 < x_2 < \cdots < \alpha(x_0).$$

Hence, it follows that the sequence x_n converge monotonically to a certain limit α:

$$x_n \uparrow \alpha \leqq \alpha(x_0).$$

Therefore, it follows from (2.1) that

$$\lim_{x_n \to \alpha} K(x_n) = 0.$$

Furthermore, if $\alpha < \alpha(x_0)$, then from (2.4), we have $-\infty < \lim_{x_n \to \alpha} X(x_n) < 1$.

Therefore it follows from Lemma 1 and $\lim_{x_n \to \alpha} h(x_n) < 0$ that we have

$$\lim_{x_n \to \alpha} K(x_n) < 0.$$

We have our contradiction. Consequently, we have $\alpha \equiv \alpha(x_0)$.

In the same way, taking x_0 such that $\zeta_0 < x_0 < \zeta_0'$, we can prove that the sequence x_n converge monotonically to $\alpha(x_0)$. Then, Theorem 1 is completely proved.

Next, let $f(x)$ be given by the form (1.2). Also in this case, by repetition of the same discussion as the above, we can show the monotonic convergence of the rule (2.1). We have:

Theorem 2. Let $f(x)$ be given by the form (1.2). Then, if $-\sqrt{b} < a \leqq 0$ and if we take the real starting value x_0 in (2.1) such that $f'(x_0)f(x_0) \neq 0$, the sequence x_n in (2.1) converge monotonically to $\alpha(x_0)$ (see Ref. 1), pp. 110–113).

Thus, it was shown that the iterative methods (2.1) have the monotonic global convergence under the same assumptions as those on Ostrowski's method for the starting value x_0.

Remarks. For $b=1$, (2.1) coinsides with Hansen and Patrick's methods.

Furthermore, for $a=0$ and $b=0$, (2.1) coincides with Ostrowski's method.

3. Modification for Multiple Zeros

In Ref. 6), for the case where α is a zero of $f(x)$ of multiplicity $m>1$, we considered the order of convergence for Eq. (1.4). Then, we showed that if $R\left(1-\frac{1}{m}\right) \neq m$, then the convergence of x_n to α is only linear. Also in this case, it was shown that we can still have cubic convergence to α by modifying the rule (1.4) in the form:

$$x_{n+1} = \phi_m(x_n),\ n=0, 1, 2, \cdots \qquad (3.1)$$

where $\phi_m(x) = x - mhR(1-m+mX)$.

Furthermore, the asymptotic error constant of $\phi_m(x)$, C_m was given by

$$C_m = \lim_{x \to \alpha} \frac{\phi_m(x)-\alpha}{(x-\alpha)^3} = \frac{1}{m^2(m+1)^2}\left\{\frac{3}{2}+\frac{m}{2} - 2R''(0)\right\}\left\{\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right\}^2 - \frac{1}{m(m+1)(m+2)} \cdot \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \qquad (3.2)$$

In our case, it follows from Lemma 1 that

$$R\left(1-\frac{1}{m}\right) \leqq \frac{1}{\sqrt{1-\left(1-\frac{1}{m}\right)}} = \sqrt{m} < m.$$

Therefore, the convergence of x_n in (2.1) to α is only linear.

Modifying (2.1) by (3.1), we have

$$x_{n+1} = x_n - mh_nR(1-m+mX_n),\quad n=0, 1, 2, \cdots \qquad (3.3)$$

where $h_n \equiv h(x_n)$, $X_n(x_n) \equiv X_n$, and $R(X) = \dfrac{a+\sqrt{b}}{a+\sqrt{b}-\sqrt{b}\,(a+\sqrt{b})X}$.

Then, from (3.2), the asymptotic error constant C_m is given by

$$C_m = \frac{1}{m^2(m+1)^2}\left(\frac{m}{2}-\frac{a}{2\sqrt{b}}\right)\left\{\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right\}^2 - \frac{1}{m(m+1)(m+2)} \cdot \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \qquad (3.4)$$

Then, we have:

Theorem 3. Let $f(x)$ be given by the form (1.3). Then, if under the conditions of Theorem 1 the muliplicity of $\alpha = \alpha(x_0)$ is m, the sequence x_n in (3.3) converge monotonically to $\alpha(x_0)$.

Furthermore, we have:

Theorem 4. Let $f(x)$ be given by the form (1.2). Then, if under the conditions of Theorem 2 the multiplicity of $\alpha = \alpha(x_0)$ is m, the sequence x_n in (3.3) converge monotonically to $\alpha(x_0)$.

In the case where α is a zero of $f(x)$ of multiplicity $m>1$, we have

$$1 - X - \frac{mh^2}{(x-\alpha)^2} = h^2\left\{\frac{P}{x^2} + 2c + \sum_k \frac{1}{(x-\alpha_k)^2} - \frac{m}{(x-\alpha)^2}\right\} > 0. \qquad (3.5)$$

Therefore, from (3.5), we have

$$|x-\alpha(x)|>\frac{\sqrt{m}\,|h|}{\sqrt{1-X}}. \quad (3.6)$$

Furthermore, since $1-m+mX<1$, using Lemma 1, we have

$$\frac{\sqrt{m}}{\sqrt{1-X}} \geqq mR(1-m+mX). \quad (3.7)$$

Hence it follows from (3.6) and (3.7) that

$$|X-\alpha(x)|>m|h|R(1-m+mX). \quad (3.8)$$

Finally, those Theorems can be shown in the way similar to the proof of Theorem 1 and Theorem 2.

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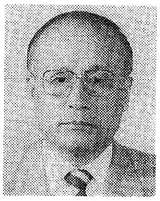

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