Induced minor free graphs: Isomorphism and clique-width

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Abstract: Given two graphs G and H, we say that G contains H as an induced minor a graph isomorphic to H can be obtained from G by a sequence of vertex deletions and edge contractions. We study the complexity of Graph Isomorphism on graphs that exclude a fixed graph as an induced minor. More precisely, we determine for every graph H that Graph Isomorphism is polynomial-time solvable on H-induced-minor-free graphs or that it is isomorphism complete. Additionally, we classify those graphs H for which H-induced-minor-free graphs have bounded clique-width. Those two results complement similar dichotomies for graphs that exclude a fixed graph as an induced subgraph, minor or subgraph.

1. Introduction

Remaining unresolved, the algorithmic problem Graph Isomorphism persists as a fundamental graph theoretic challenge which, despite generating ongoing interest, has neither been shown to be NP-hard nor polynomial-time solvable. The problem asks whether two given graphs are structurally the same, that is, whether there exists an adjacency and non-adjacency preserving map from the vertices of one graph to the vertices of another graph.

Related work. In the absence of a result determining the complexity of the general problem, considerable effort has been put into classifying the isomorphism problem of graph classes as being polynomial time tractable or polynomial time equivalent to the general problem, i.e., GI-complete. Most graph classes considered in these efforts are graph classes that are closed under some basic operations. Operations that are typically considered are edge contraction, vertex deletion and edge deletion. A class of graphs closed under all of these operations is said to be minor closed and can also be described as a class of graphs avoiding a set of forbidden minors. As shown by Ponomarenko, the Graph Isomorphism problem can be solved in polynomial time on H-minor free graphs for any fixed graph H [22]. This implies prior results on solvability of graphs of bounded treewidth, planar graphs and bounded genus. The result on minor closed graph classes was recently extended by Grohe and Marx to H-topological minor free graphs [11], and Lokshtanov, Pilipczuk, Pilipczuk and Saurabh [17] showed that the problem is actually FPT on graphs of bounded treewidth, an important class of minor-free graphs. When a graph class is only required to be closed under some of the above named operations, isomorphism on such a graph class can sometimes be polynomial-time solvable and sometimes be isomorphism complete. We say that a graph G is H-free if it does not contain the graph H as an induced subgraph. When forbidding one induced subgraph, it is known that Graph Isomorphism can be solved in polynomial time on H-free graphs if H is an induced subgraph of P_4 (the path on 4 vertices) and GI-complete otherwise (see [2]). For two forbidden induced subgraphs such a classification into graph isomorphism complete and polynomial-time solvable cases turns out to be more complicated [16], [23]. In case we consider forbidden subgraphs (i.e., also allowing edge and vertex deletions) there is a complete dichotomy for the computational complexity of Graph Isomorphism on classes characterized by a finite set of forbidden subgraphs, while there are intermediate classes defined by infinitely many forbidden subgraphs [18] (assuming that graph isomorphism is not polynomial time solvable).

Our results. In this paper we consider graph classes closed under edge contraction and vertex deletion (but not necessarily edge deletion). The corresponding graph containment relation is called induced minor. More precisely, a graph H is an *induced minor* of a graph G if H is obtained from G by repeated vertex deletion and edge contraction. If no induced minor of G is isomorphic to H, we say that G is H-induced-minor-free. We consider graph classes characterized by one forbidden induced minor, and on those classes we study the computational complexity of the Graph Isomorphism problem and whether the value of the parameter clique-width is bounded by some universal constant c_H. The isomorphism problem for such classes was first considered by Ponomarenko [22] for the case where H is connected. In that paper two choices for the graph H play a crucial role, namely choosing H to be the gem and choosing H to be co-$(P_3 \cup 2K_1)$ (see Figure 1). Forbidding either of these graphs as induced minor yields a graph class with an isomorphism problem solvable in polynomial time. However, to show polynomial time solvability for the gem, the proof of [22], due to a misunderstanding concerning the required preconditions, incorrectly relies on a technique of [13] to reduce the problem to the 3-connected case (see Subsection 3.2). We provide a proof that avoids this reduction and

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instead use a reduction of the problem to the 2-connected case for which we provide a polynomial time isomorphism test. To extend Ponomarenko's theorem to the disconnected case, we provide a reduction structurally different from the ones used previously, allowing us to treat the case where H consist of a cycle with an added isolated vertex. Overall we extend Ponomarenko's results to obtain the following theorem (see Figure 1 for the graphs that are mentioned).

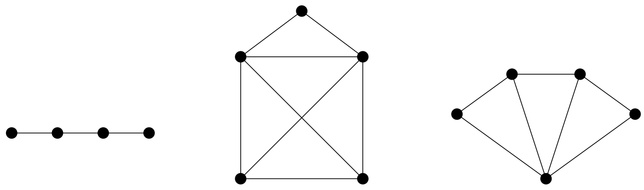

Fig. 1 The small graphs used in our main theorems.

Theorem 1.1. *Let H be a graph. The* Graph Isomorphism *problem on H-induced-minor-free graphs is polynomial-time solvable if H is complete or an induced subgraph of P_4, co-$(P_3 \cup 2K_1)$ or the gem, and GI-complete otherwise.*

Our proofs rely on structural descriptions that also allow us to determine exactly which classes characterized by one forbidden induced minor have bounded clique-width.

Theorem 1.2. *Let H be a graph. The clique-width of the H-induced-minor-free graphs is bounded if and only if H is an induced subgraph of P_4, co-$(P_3 \cup 2K_1)$, or the gem.*

While it is still open whether Graph Isomorphism is polynomial time solvable for graph of bounded clique-width, our theorems are in accordance with the seemingly reoccurring pattern that the isomorphism problem for graphs of bounded clique-width is polynomial time solvable, while there are graph classes with unbounded clique width on which Graph Isomorphism is polynomial-time solvable. Additionally, note that H-free graphs have bounded clique-width if and only H is an induced subgraph of P_4 and that H-minor-free graphs have bounded clique-width if and only if H is planar. Recently, Paulusma and Dabrowski gave a dichotomy for the clique-width of bipartite H-free graphs [7], and initiated the study of clique-width on graphs that forbid two graphs as induced subgraphs [8].

Structure of the paper. We first summarize well known observations about induced-minor-free graphs, isomorphism and clique-width (Section 2). We then consider classes that are characterized by one forbidden induced minor of size at most 5 (Section 3). Finally we show that the observations of Sections 2 and 3 resolve all cases with forbidden induced minors of size at least 6 (Section 4). In this paper all graphs that are considered are finite. Throughout the paper, we use standard notation and terminology from Diestel [10].

2. Basic observations

In this section, we summarize a few well-known basic observations about graph classes closed under induced minors and clique-width.

2.1 Clique-width

In [6], Courcelle and Olariu introduced the clique-width of graphs as a way of measuring the complexity of minimal separators in a graph. Similarly to graphs of bounded treewidth, it has been shown that a large class of problems can be solved efficiently on graphs of bounded clique-width [5]. However, while Graph Isomorphism has long been known to be solvable in polynomial time on graphs of bounded treewidth [22], it is not currently known whether the problem is tractable on graphs of bounded clique-width.

For any given graph G, the clique-width of G, denoted by $\mathrm{cw}(G)$, is defined as the minimum number of labels needed to construct G by means of the following 4 operations: (i) Creation of a new vertex v with label i (denoted $i(v)$); (ii) Forming the disjoint union of two labeled graphs G_1 and G_2; (iii) Joining by an edge every vertex labeled i to every vertex labeled j, where $i \neq j$; (iv) Renaming label i to label j. In the remainder of the paper, we will be using the following well-known observations to derive upper and lower bounds on the value of clique-width of H-induced-minor-free graphs. See e.g., [12] for an overview of clique-width.

Theorem 2.1 ([6]). *Let G be a graph and $\overline{G}$ its edge complement, then* $\mathrm{cw}(G) \leq 2 \cdot \mathrm{cw}(\overline{G})$.

Observation 2.2. *Let G be a graph and S a subset of vertices of G. We have* $\mathrm{cw}(G \setminus S) \leq \mathrm{cw}(G) \leq 2^{\mathrm{cw}(G \setminus S)+|S|+1} - 1$.

Let G be a graph and u a vertex of G. The *local complementation* of G at u is the graph obtained from G by replacing the subgraph induced by the neighbors of G with its edge complement. The following observation follows from the well-known facts that for any graph G, we have $\mathrm{cw}(G) \leq 2^{\mathrm{rw}(G)+1} - 1$ (see [21]), where rw denotes the rank-width, and that rank-width remains constant under local complementations [20].

Observation 2.3. *Let G and G' be two graphs such that G' can be obtained from G by a sequence of local complementations, then* $\mathrm{cw}(G) \leq 2^{\mathrm{cw}(G')+1} - 1$.

Theorem 2.4 ([4]). *Let G and G' be two graphs such that G' can be obtained from G by a sequence of edge subdivisions, i.e., replacing edges with paths of length 2. Then* $\mathrm{cw}(G) \leq 2^{\mathrm{cw}(G')+1} - 1$.

Observation 2.5 ([1]). *Let G be a graph and $\mathcal{B}$ the set of its biconnected components. It holds that* $\mathrm{cw}(G) \leq 2^{t+1} - 1$, *where* $t = \max_{B \in \mathcal{B}}\{\mathrm{cw}(B)\}$.

Finally, note that for any graph G, the clique-width of G is at most $3 \cdot 2^{\mathrm{tw}(G)-1}$, where $\mathrm{tw}(G)$ denotes the treewidth of G [3].

2.2 Some tractable cases

Lemma 2.6. *If H is a complete graph, then* Graph Isomorphism *for H-induced-minor-free graphs can be solved in polynomial time.*

Lemma 2.7. *Let H be a complete graph K_k. The H-induced-minor-free graphs have bounded clique-width if and only if $k \leq 4$.*

Note that the lemma above is used to prove Theorem 1.2, but K_4 is not explicitly mentioned in the statement, due to the fact that K_4 is an induced subgraph of co-$(P_3 \cup 2K_2)$.

Lemma 2.8. *If H is an induced subgraph of P_4 then* Graph Isomorphism *for the H-induced-minor-free graphs can be solved in linear time.*

It is well known that P_4-free graphs are exactly the graphs of clique-width at most 2 (see [14]).

2.3 Some intractable cases

A *split partition* (C, I) of a graph G is a partition of $V(G)$ into a clique C and an independent set I. A *split graph* is a graph admitting a split partition. A split graph is of *restricted split type* if it has a split partition (C, I) such that each vertex in I has at most two neighbors in C. Note that a non-complete split graph of restricted split type has minimum degree at most 2. The classes of co-bipartite graphs and restricted split graphs are closed under vertex deletions and edge contractions, and thus under induced minors. As also argued in [22] and [16], the standard graph-isomorphism reductions to split graphs and co-bipartite graphs explained in [2] imply the following lemmas.

Lemma 2.9. *If H is not of restricted split type and co-bipartite, then* Graph Isomorphism *for the H-induced-minor-free graphs is GI-complete.*

The reductions used in the lemma can be achieved by performing edge subdivisions and subgraph complementation. *Subgraph complementation* is the operation of complementing the edges of an induced subgraph. The clique-width of graphs in the class obtained by applying subgraph complementation a constant number of times is bounded if and only if it is bounded for graphs in the original class [14]. Together with Theorem 2.4, this implies that restricted split graphs and co-bipartite graphs obtained by the reductions from general graphs have unbounded clique-width.

Corollary 2.10. *If H is not of restricted split type and co-bipartite, then the H-induced-minor-free graphs have unbounded clique-width.*

3. Graphs on at most 5 vertices

In this section we study graph classes characterized by a forbidden induced subgraph H that has at most 5 vertices.

3.1 The graph $K_3 \cup K_1$

We show that Graph Isomorphism is GI-complete on graphs that do not contain $K_3 \cup K_1$ as an induced minor. Additionally, we show that these graphs have unbounded clique-width.

Theorem 3.1. *The* Graph Isomorphism *problem is isomorphism complete on graphs that do not contain $K_3 \cup K_1$ as an induced minor.*

We now prove that $K_3 \cup K_1$-induced-minor-free graphs do not have bounded clique-width.

Theorem 3.2. *The class of graphs that do not contain $K_3 \cup K_1$ as an induced minor does not have bounded clique-width.*

3.2 The gem

We now consider the class of graphs that do not contain the gem as an induced subgraph (see **Fig. 1**). In [22] this class is also considered, however, there is an issue with the proof for the fact that the isomorphism problem of graph in this class is polynomial-time solvable. More precisely, a common misunderstanding of how the reduction to three connected components by Hopcroft and Tarjan [13] is to be applied has happened. Indeed, the techniques of Hopcroft and Tarjan do not show that graph isomorphism in a graph class C polynomial-time reduces to graph isomorphism of 3-connected components in C, even if C is a minor closed graph class. If this were the case then the class of split graphs of restricted type would be polynomial-time solvable since the only 3-connected graphs of this type are complete graphs. Additionally to C being minor closed, for the techniques to be applicable it is necessary to solve the edge-colored problem for 3-connected graphs in C. However, edge-colored isomorphism is already GI-complete on complete graphs.

We now provide a proof that isomorphism of graphs not containing the gem as an induced subgraph is polynomial-time solvable without alluding to 3-connectivity. For this we first need to extend the structural considerations for such graphs performed in for 3-connected graphs [22] to biconnected graphs.

Let C be a subgraph of G. We say a vertex v in a vertex set $M \subseteq V(G) \setminus C$ has *exclusive attachment* with respect to C among the vertices of M if $N(v) \cap C \neq \emptyset$ but there is no vertex $v' \in M \setminus \{v\}$ with $(N(v) \cap C) \cap (N(v') \cap C) \neq \emptyset$. That is, no other vertex of M shares a neighbor in C with v.

Lemma 3.3. *Let G be a biconnected gem-induced-minor-free graph. Suppose C is a biconnected subgraph of G with at least 3 vertices and M is a component of $G - C$ such that $N(M) \cap C \neq C$. If $v \in M$ is a vertex with $|N(v) \cap C| = 1$ then v has exclusive attachment.*

Lemma 3.4. *Let G be a biconnected gem-induced-minor-free graph. Suppose C is a biconnected subgraph of G and M is a component of $G - C$ with $N(M) \cap C \neq C$ and $|N(M) \cap C| \leq 3$. If there is no vertex x in M with $|N(x) \cap C| = 1$ then every vertex of M has a neighbor in C, and M is a P_4-free graph.*

We call a vertex of a biconnected graph G a *branching vertex* if it has degree at least 3.

Lemma 3.5. *Let G be a biconnected gem-induced-minor-free graph that contains the path P_4 as an induced subgraph. Then at least one of the following two options holds:*

- *G has a subgraph H which is an induced path containing at most 2 inner vertices that are branching vertices of G such that $G - H$ is disconnected, or*
- *G has a subgraph H that is a cycle containing at most 3 branching vertices of G such that for every connected component M of $G - H$ we have $N(M) \cap H \neq H$.*

Let G be a graph with subgraphs H and K. We say that G is *sutured* from H and K along $V \subseteq V(H)$ and $V' \subseteq V(K)$ if G is obtained in the following way. First we require that $|V| = |V'|$. We also require that $V(H) \cap V(K) = V \cap V'$. The graph G must then be formed from $K \cup H$ in the following way. We add edges that form a perfect matching between vertices in $V \setminus V'$ and $V' \setminus V$. Finally we may subdivide the edges in the matching an arbitrary number of times (See Figure 2).

Lemma 3.6. *Let G be a biconnected gem-induced-minor-free graph. There exists an induced subgraph H of G which is isomorphic to either a path or a cycle, contains at most 4 branching vertices, and such that for every component M of $G - H$ the following holds: the graph $G[M \cup H]$ is sutured from H and some graph K along V and V' such that $K \setminus V'$ is P_4-free. Moreover $|V'| \leq 4$ and every vertex of $K - V'$ has a neighbor in V'.*

Theorem 3.7. *The* Graph Isomorphism *problem can be solved in polynomial time on gem-induced-minor-free graphs.*

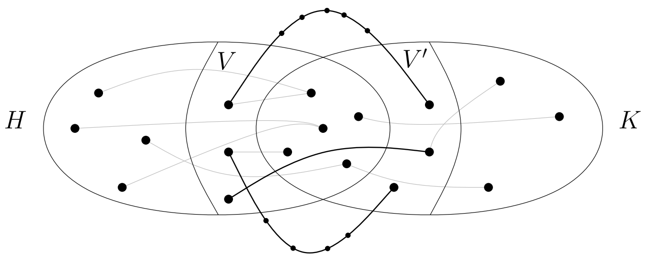

Fig. 2 A suture of two graphs H and K.

Proof. It is folklore that graph isomorphism of a hereditary graph class C reduces to isomorphism of vertex-colored biconnected graphs in C (see for example [9] or [19]). We thus assume that the input graphs are colored and biconnected. If G is such a biconnected graph, we search for a subgraph H that satisfies the assumptions of Lemma 3.6, that is, H is a path or a cycle with at most 4 branching vertices such that for every component M of $G - H$ we know that $G[M \cup H]$ is a suture of H with a graph K such that $K \setminus V'$ is P_4-free, where V' are the attachments in K. Moreover $|V'| \leq 4$ and every vertex in $K - V'$ has a neighbor in V'. Each H is determined by the branch vertices, the leaves (if H is a path) and choices of the paths of non-branching vertices connecting such vertices. Suppose now G_1 and G_2 are biconnected input graphs to the isomorphism problem. Since there are only polynomially many possible choices for H, we can find a graph H_1 in G_1 with said properties and test for every H_2 in G_2 whether there is an isomorphism that maps H_1 to H_2. To do so we iterate over all isomorphism φ from H_1 to H_2, there are only polynomially many, and check whether such an isomorphism extends to an isomorphism from G_1 to G_2. To check whether such an isomorphism extends, it suffices to know which component M_1 of $G_1 - H_1$ can be mapped isomorphically to which component M_2 of $G_2 - H_2$ such that the isomorphism can be extended to an isomorphism from $G_1[H_1 \cup M_1]$ to $G_2[H_2 \cup M_2]$ such that H_1 is mapped to H_2 in agreement with φ.

Note that the mapping φ determines how vertices with exclusive attachment in H_1 must be mapped. Repeatedly increasing H_1 by these vertices and H_2 by their images we end up at a part of M_1 that is P_4-free and adjacent to at most 3 vertices to which images have already been determined.

The isomorphism problem for vertex-colored P_4-free is solvable in polynomial time (see [23]) and thus the problem for graphs obtained from P_4-free by adding a bounded number of vertices can be solved in polynomial time ([15], Theorem 1). Using this algorithm the theorem follows. □

Theorem 3.8. *If H is an induced subgraph of the gem, then the H-induced-minor-free graphs have bounded clique-width.*

3.3 The graph co-($P_3 \cup 2K_1$)

In the following we will analyze the graphs that do not contain an induced minor isomorphic to co-($P_3 \cup 2K_1$), the graph obtained from K_5 by removing two incident edges. While it has already been shown in [22] that isomorphism for such graphs reduces to isomorphism of graphs not containing the gem (and is thus polynomially solvable), we provide refinement of the proof in [22] for this. We do this to obtain a finer structural description of the graphs, allowing us also to bound the clique-width in the graph class.

Suppose G is a co-($P_3 \cup 2K_1$)-induced-minor-free graph. If G does not have a K_t minor for some fixed t then G is in particular in the minor closed graph class of K_t-minor free graphs, and, as described in the introduction, the isomorphism problem can be solved in polynomial time for such graphs. Our strategy is thus to find a K_t minor and use this to analyze the structure of G. In general, of course, there is no constant bound on the number of vertices required to form a K_t minor. However in a co-($P_3 \cup 2K_1$)-induced-minor-free graph there is such a bound. We call a K_t minor *compact* if every bag has at most 2 vertices.

Lemma 3.9. *If a co-($P_3 \cup 2K_1$)-induced-minor-free graph G has a K_t minor for $t \geq 5$ then G has a compact K_t minor.*

Proof. Let $M_1, \ldots, M_t$ be the bags of a K_t minor in G such that the M_i are inclusion minimal with respect to forming a K_t minor. That is, removing a vertex from one of the M_i yields a minor different from K_t. We analyze the structure of the minor. We say a vertex v is adjacent to a bag M_j if there exists a vertex $v' \in M_j$ that is adjacent to v.

For a vertex $v \in M_i$ define $\text{Mdeg}(v) = |\{M_j \mid j \neq i, N(v) \cap M_j \neq \emptyset\}|$ to be the number of bags different from M_i adjacent to v. Using several steps we will show that $\text{Mdeg}(v) \geq t - 2$ for all $v \in M_1 \cup M_2 \cup \cdots \cup M_t$. We first argue that in case $\text{Mdeg}(v) > 1$ then $\text{Mdeg}(v) \geq t - 2$. Indeed, if $\text{Mdeg}(v) > 1$ then consider the minor obtained by removing all vertices from M_i different from v. If $\text{Mdeg}(v) < t - 2$ we can choose 2 bags which have vertices adjacent to v and two bags which do not have such vertices. Using these bags and the vertex v we obtain the forbidden induced minor co-($P_3 \cup 2K_1$). We call vertices with $\text{Mdeg}(v) = 0$ inner vertices, those with $\text{Mdeg}(v) = 1$ low degree vertices and we call vertices with $\text{Mdeg}(v) \geq t - 2$ high degree vertices. Next we argue that there are at most 2 high degree vertices in each bag. Indeed, if there are at least 2 such vertices, we can pick two high degree vertices v, v' in M_i which are not adjacent to exactly the same bags such that there is a path from v to v' in M_i that does not contain any other high degree vertex. Since every bag different from M_i is adjacent to v or v', removing all vertices different from v and v' and not lying on the path yields a K_t minor. Since the bags $M_1, \ldots, M_t$ were chosen to be minimal, we conclude that there are at most 2 high degree vertices in each bag.

We further argue that there is no low degree vertex in M_i. Indeed, in case there is at least one low degree vertex in M_i, we can choose a low degree vertex $v \in M_i$ and a vertex $v' \in M_i$ adjacent to a bag M_j with $j \neq i$ such that v is not adjacent to M_j and such that there exists a path in M_i of inner vertices connecting v and v'. We remove all vertices in M_i different from v and v' and not on said path connecting them. We then move the vertex v' from M_i to M_j. We obtain the induced minor co-($K_{1,t-3} \cup 2K_1$), which contains co-($P_3 \cup 2K_1$) since $t > 2$.

Finally we argue that there are no inner vertices. Indeed, by minimality we can assume that every inner vertex v lies on a path between two high degree vertices v_1 and v_2, say. We again remove all vertices different from v_1 and v_2 not on the path. We then move v_1 to an adjacent bag M_j and v_2 to an adjacent bag $M_{j'}$ such that $j \neq j'$. This is possible since the vertices have high degree. Again we obtain a forbidden induced minor co-$(K_{1,t-3} \cup 2K_1)$ as above.

Since there are only high degree vertices and since each bag can only contain two such vertices, the minimal minor is compact. □

Lemma 3.10. *If G is a biconnected co-$(P_3 \cup 2K_1)$ induced-minor-free graph and M is a compact K_t minor with $t \geq 5$ then $G - M$ is $(K_2 \cup K_1)$-free.*

Corollary 3.11. *If a biconnected co-$(P_3 \cup 2K_1)$-induced-minor-free graph G has a K_8 minor then G is $(K_2 \cup K_1)$-free.*

Since the gem is biconnected, and thus every occurrence of a gem as induced minor must occur within a biconnected component of a graph, the corollary is a refinement of Ponomareko's result [22] that says that if a co-$(P_3 \cup 2K_1)$-induced-minor-free graph G has a $K_{2^{18}+4}$-minor then it does not contain a gem as induced minor.

Theorem 3.12. *Graph isomorphism for co-$(P_3 \cup 2K_1)$-induced-minor-free graphs can be solved in polynomial time.*

To show that the co-$(P_3 \cup 2K_1)$-induced-minor-free graphs have bounded clique-width, we need the following fact, which was indirectly proven by van 't Hof et al. in the proof of Theorem 9 in [24].

Theorem 3.13 ([24], Proof of Theorem 9)**.** *For any graph F and for any planar graph H, there exists a constant $c_{F,H}$ such that an F-minor-free graph of treewidth at least $c_{F,H}$ has H as an induced minor.*

Theorem 3.14. *If H is an induced subgraph of co-$(P_3 \cup 2K_1)$, then the H-induced-minor-free graphs have bounded clique-width.*

3.4 The remaining graphs on at most 5 vertices

Now we study the remaining small graphs of at most five vertices. We show that every case here can be reduced to some case we have solved already.

Lemma 3.15. *Let H be a non-complete graph of 5 vertices. If H is neither co-$(P_3 \cup 2K_1)$ nor the gem, then* Graph Isomorphism *for the H-induced-minor-free graphs is GI-complete.*

Lemma 3.16. *Let H be a graph of at most 4 vertices. The* Graph Isomorphism *problem for the H-induced-minor-free graphs is polynomial-time solvable if H is an induced subgraph of either co-$(P_3 \cup 2K_1)$ or P_4. Otherwise, it is GI-complete.*

The two lemmas above together imply the following theorem.

Theorem 3.17. *Let H be a non-complete graph of at most 5 vertices. Then* Graph Isomorphism *for the H-induced-minor-free graphs is polynomial-time solvable if H is an induced subgraph of P_4, co-$(P_3 \cup 2K_1)$, or the gem; otherwise, it is GI-complete.*

The reductions we used above in order to show the GI-completeness preserve the property that the clique-width is unbounded (see Subsection 2.3). Thus we have the following corollary.

Corollary 3.18. *Let H be a non-complete graph of at most 5 vertices. Then the H-induced-minor-free graphs have bounded clique-width if and only if H is an induced subgraph of P_4, co-$(P_3 \cup 2K_1)$, or the gem.*

4. Non-complete graphs on at least 6 vertices

In this section, we show that if H is not a complete graph and has at least six vertices, then Graph Isomorphism for the H-induced-minor-free graphs is GI-complete.

Lemma 4.1. *If H is non-complete and contains a clique of size 5, then* Graph Isomorphism *for the H-induced-minor-free graphs is GI-complete.*

Theorem 4.2. *If H is a non-complete graph of size at least 6, then* Graph Isomorphism *for the H-induced-minor-free graphs is GI-complete.*

Since the reductions that we used above in order to show the GI-completeness preserve the property that the clique-width is unbounded (see Subsection 2.3), we have the following corollary.

Corollary 4.3. *If H is a non-complete graph of size at least 6, then the H-induced-minor-free graphs have unbounded clique-width.*

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