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Regular Paper

Pentadral Complices

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Abstract: A parallelohedron is a convex polyhedron which fills the space by translations only. There are five families of parallelohedra. A pentadron is a pentahedron whose copies compose at least one member of every family of parallelohedra. A pentadral complex is a convex polyhedron which is constructed by combining copies of pentadra in a face-to-face gluing manner. In this paper, reversibilities and tessellabilities of pentadral complices and their related topics are studied.

Keywords: pentadron, parallelohedron, pentadral complex, reversibility, tessellability, seed of a space-filler

1. Introduction

A **pentadron** is a convex pentahedron one of whose nets is as in **Fig. 1**. Note that there is a pair of **'male'** pentadron and **'female'** pentadron which are mirror image of each other. A **pentadral complex**, or simply **pc**, is a convex polyhedron which is constructed by copies of pentadron in a face-to-face gluing manner. In a pentadral complex, we do not distinguish male and female pentadra, i.e., a pc may include both male and female pentadra (**Fig. 2**).

A **parallelohedron** is a polyhedron which fills the space by translations only. There are five families of parallelohedra, namely, parallelepiped, hexagonal prism, truncated octahedron, rhombic dodecahedron, elongated rhombic dodecahedron [1], [2], denoted by $\boldsymbol{F_i}$ $(i = 1, 2, 3, 4, 5)$, respectively (**Fig. 3**). An **affine stretching transformation** is a transformation, including affine transformation, which preserve parallelism of sides. The following theorem is proved in Ref. [3].

Theorem A For all parallelohedra P in a family F_i $(i = 1, 2, 3, 4, 5)$, there exists an affine stretching transformation ϕ such that $\phi(P)$ is a pentadral complex $p_i \in F_i$ (**Fig. 4**).

On the other hand, regular polyhedra (polytopes) are not composed of single polyhedron (polytope). See Refs. [4], [5] for the minimum number of elements (polyhedra or polytopes) required to construct all the regular polyhedra (polytopes).

Theorem 1 There exists a convex pentadral complex P such that P includes pcs $q_i \in F_i$ $(i = 1, 2, 3, 4, 5)$ as its subcomplices.

Proof:

An elongated rhombic dodecahedron p_5 made by 384 pentadra as in Fig. 4 includes a pc $q_i \in F_i$ for each $i = 1, 2, 3, 4, 5$ as a subcomplex (**Fig. 5**). □

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2. Reversion Problems on Pentadral Complices

Given two convex polyhedra α and β, we say that a pair α and β is **reversible**, denoted by $\alpha \sim \beta$, if α and β have dissections into a common finite number of hinged pieces which can be rearranged to form β and α respectively, under the following conditions [6]:

(1) The entire surface of one polyhedron gets into the interior of the other and

(2) The set of dissection planes of each polyhedron is connected and does not include any (part of) edge of it.

A pair of pcs α and β is called **reversible** if $\alpha \sim \beta$ and every pieces involved in the reversion are pcs.

A pc α is called **reversible** if there is a pc β such that $\alpha \sim \beta$.

A pc α is called **self-reversible** if $\alpha \sim \alpha$.

Theorem 2 There exist self-reversible pentadral complicies $f_i \in F_i$ $(i = 1, 2, 3, 4, 5)$.

Proof:

The pentadral complex $f_i \in F_i$ for each $i = 1, 2, 3, 4, 5$ as shown in **Fig. 6** is self-reversible. □

Problem 1 Make a self-reversible pentadral complex $f_i \in F_i$ for each $i = 1, 2, 3, 4, 5$ whose pieces preserve a ring-structure by piano-hinges.

It is shown in Ref. [7] that a self-reversible cube which is a pentadral complex has a ring-structure (**Fig. 7**).

Theorem 3 There exist pentadral complices $f_1, f_1', f_1'', f_1''' \in F_1$, $f_2, f_2', f_2'' \in F_2$, $f_3 \in F_3$, $f_4 \in F_4$, $f_5 \in F_5$ which satisfy $f_1 \sim f_2$, $f_1' \sim f_3$, $f_2' \sim f_3$, $f_1'' \sim f_4$, $f_1''' \sim f_5$, $f_2'' \sim f_5$.

Proof:

One can compose reversible pairs of pcs analogously to Theorem 2. □

Problem 2 It is not known whether there exists a reversible pair of pentadral complices between two families stated as dotted edges in **Fig. 8**. Determine whether there exist pentadral complices $g_2 \in F_2, g_3, g_3' \in F_3, g_4, g_4', g_4'' \in F_4, g_5, g_5' \in F_5$ which satisfies the following relations or not.

(1) $g_2 \sim g_4$,

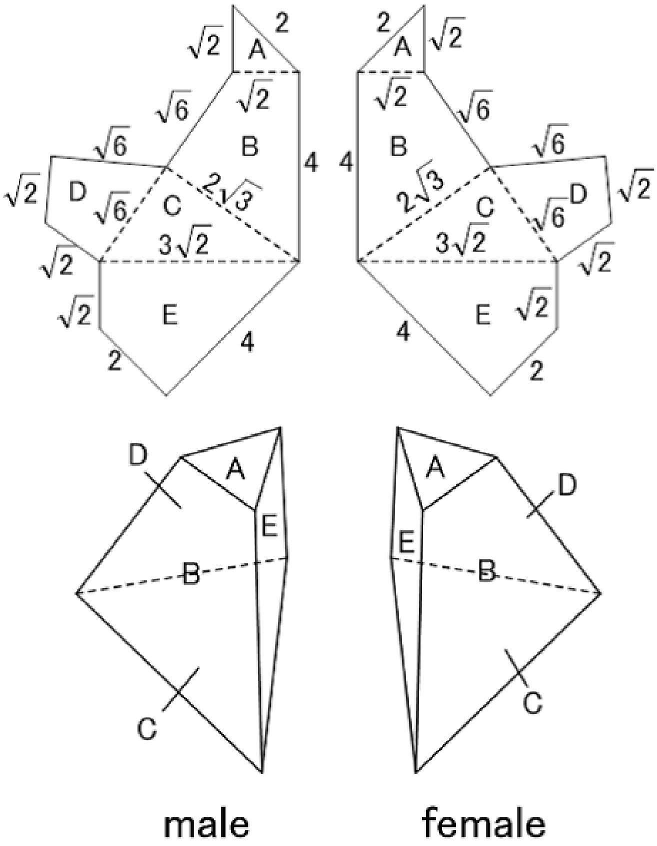

Fig. 1 A symmetric pair of pentadra and their nets.

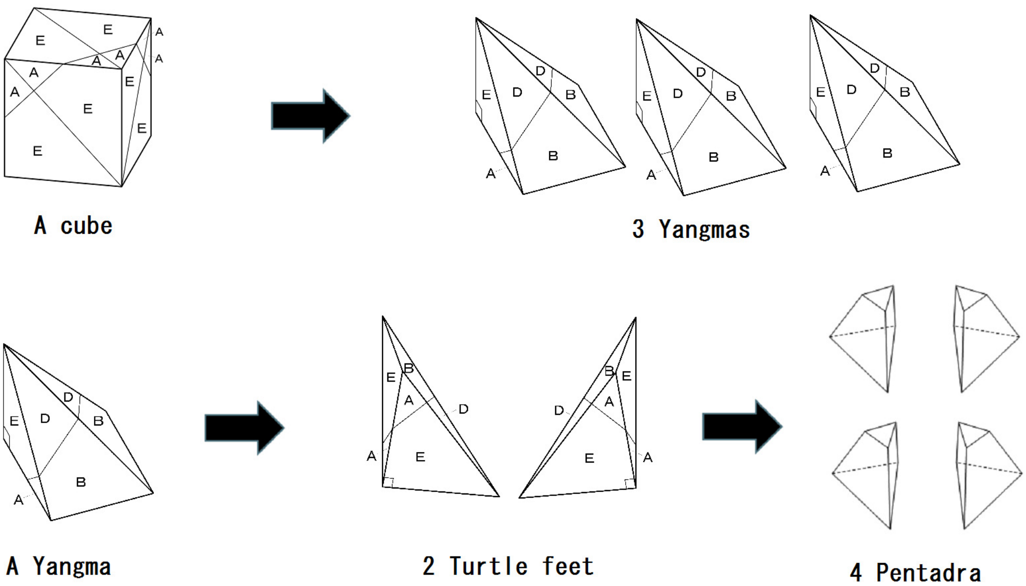

Fig. 2 A decomposition of a cube into 12 pentadra. All the polyhedra appearing in this figure are pentadral complices.

(2) $g_3 \sim g'_4$, $g_3 \sim g_5$,
(3) $g''_4 \sim g'_5$.

3. Seeds of a Space-filler

A convex polyhedron is called a **space-filler** if its copies fills the space in a face-to-face gluing manner. For a given polyhedron P, a ***P*-complex** is a convex polyhedron which is constructed by copies of P (including mirror images of P) in a face-to-face gluing manner. We say a polyhedron P is a **seed of a space-filler** (briefly **seed**) if any P-complex is a space-filler.

Problem 3 Every cuboid is trivially a seed, since any cuboidal complex is also a cuboid (**Fig. 9**). Are there any seeds other than cuboids?

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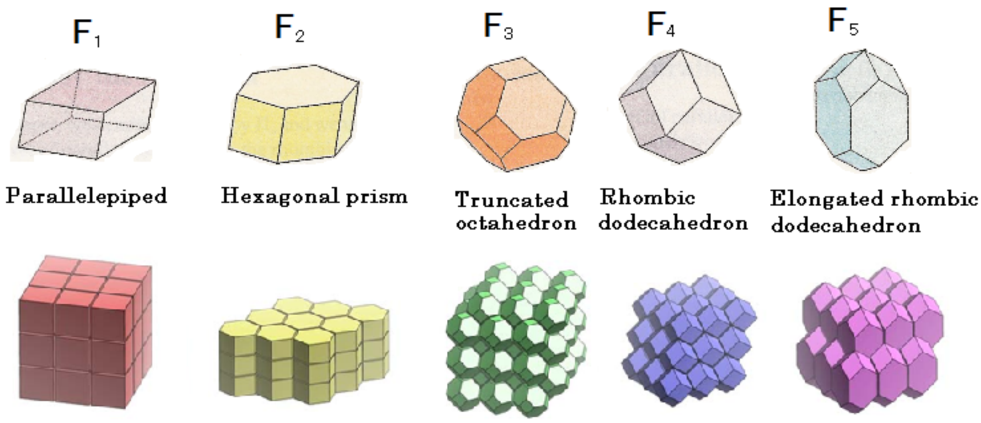

Fig. 3 The five families of parallelohedra by Fedorov.

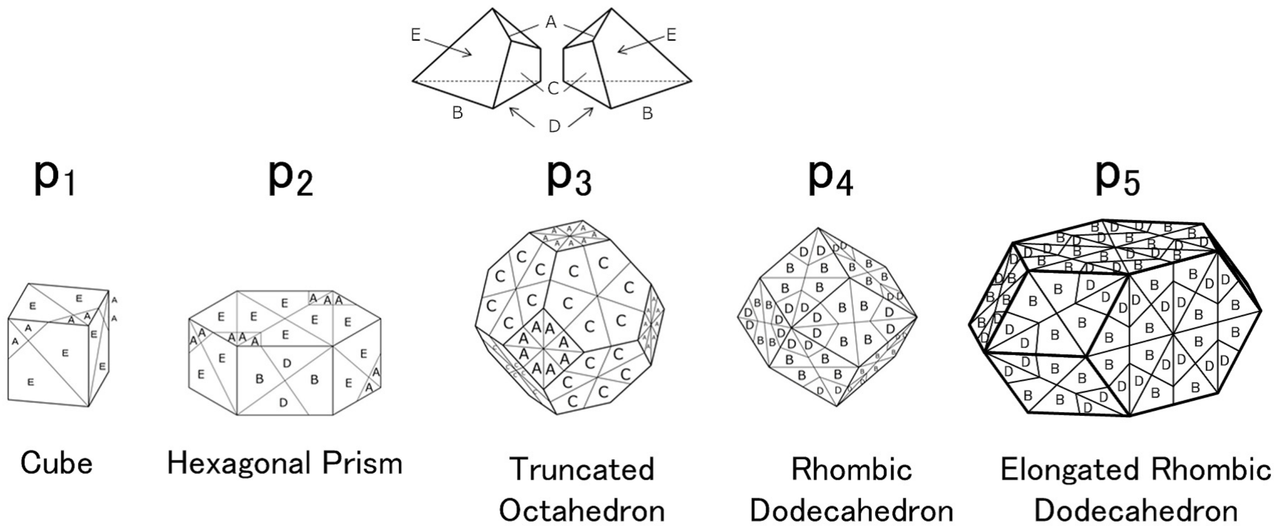

Fig. 4 Parallelohedra as pentadral complices.

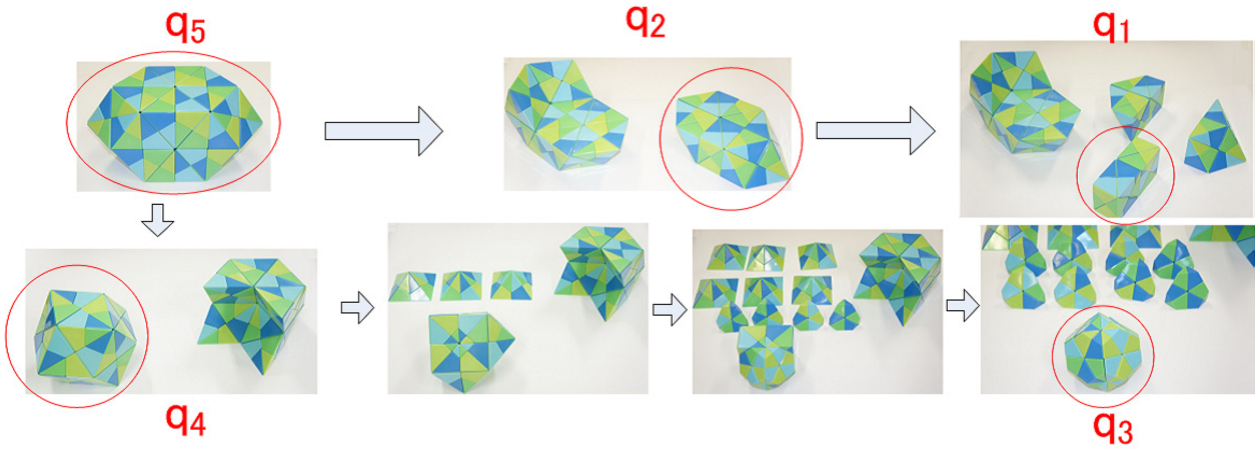

Fig. 5 An elongated rhombic dodecahedron including pcs $q_i \in F_i$ $(i = 1, 2, 3, 4, 5)$ as a subcomplex.

3.1 Pentadral Complices and Their Tessellability

Note that there is unique way to fill the space by pentadra in a face-to-face gluing manner. The tessellation by pentadra is denoted by T_P (**Fig. 10**).

A **quasi-pc** is a (possibly concave) polyhedron constructed by copies of pentadron in a face-to-face gluing manner.

Proposition 1 There exist seven combinatorially possible quasi-pcs consisting of two pentadra.

In a pentadral complex, every pair of adjacent pentadra, sharing a common face, forms either a turtle foot (when the two pentadra have the same sex) or one of four quasi-pcs $S_2^a, S_2^b, S_2^c, S_2^d$ as in **Fig. 11** (when the two pentadra have different sexes).

One quasi-pc for the same sex and one quasi-pc for different sexes does not appear in a pentadral complex (**Fig. 12**).

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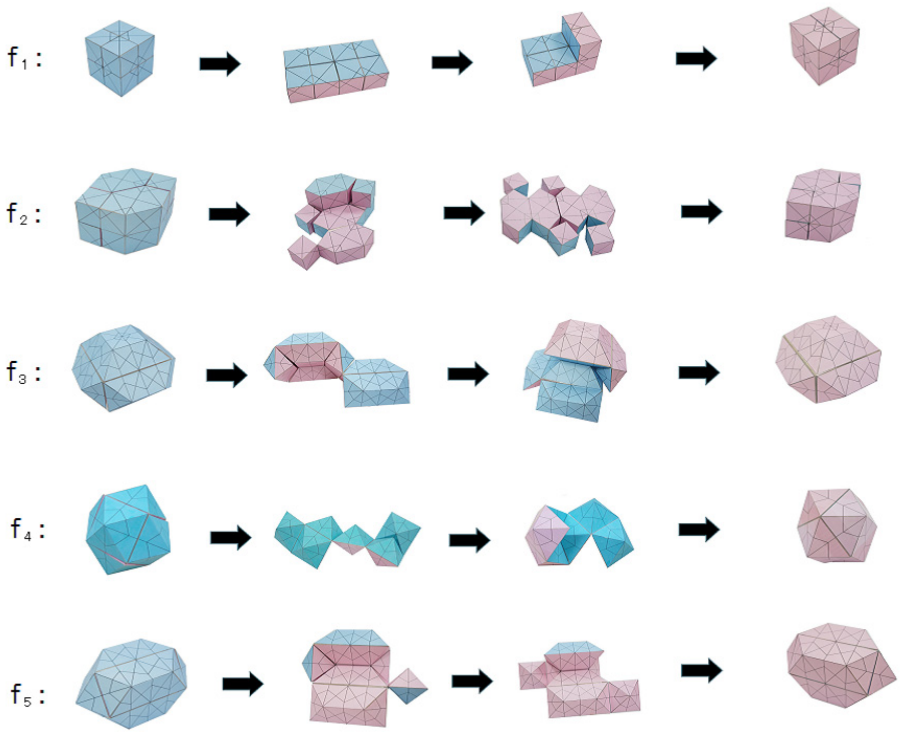

Fig. 6 Self-reversible pentadral complices $f_i \in F_i$ $(i = 1, 2, 3, 4, 5)$.

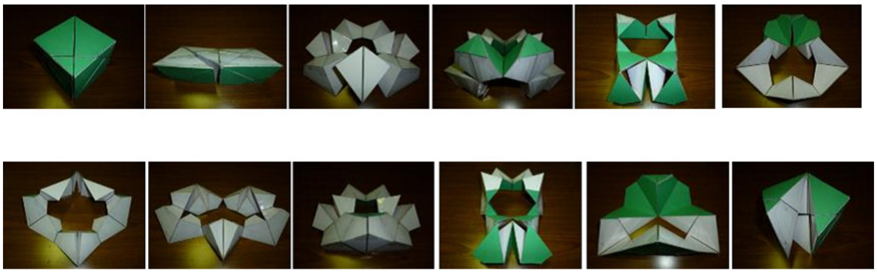

Fig. 7 Self-reversible cube which is a pentadral complex with ring-structure.

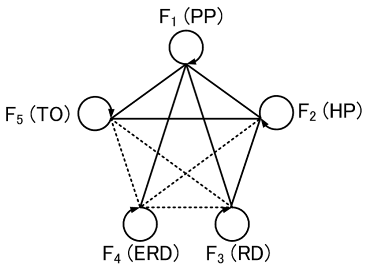

Fig. 8 Bold edges indicate that there exist reversible pairs of pentadral complices between corresponding pair of families.

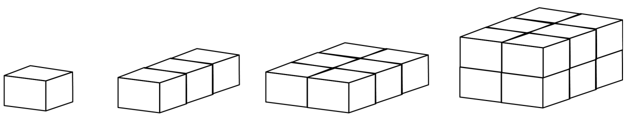

Fig. 9 An example of a seed.

Lemma 1 Every pentadral complex is a subcomplex of T_P.

Proof:

Note that all of the five types in Propositon 1 are a part of T_P. By the uniqueness of T_P, every pentadral complex is a subcomplex of T_P. □

Each pentadron in T_P belongs to exactly one turtle foot with unique partner of it. Such two pentadra with the same sex in the same turtle foot are called a **coupled pair** of pentadra.

Lemma 2 If a pentadral complex C includes one coupled pair as a subcomplex, then all the pentadra in C should be coupled,

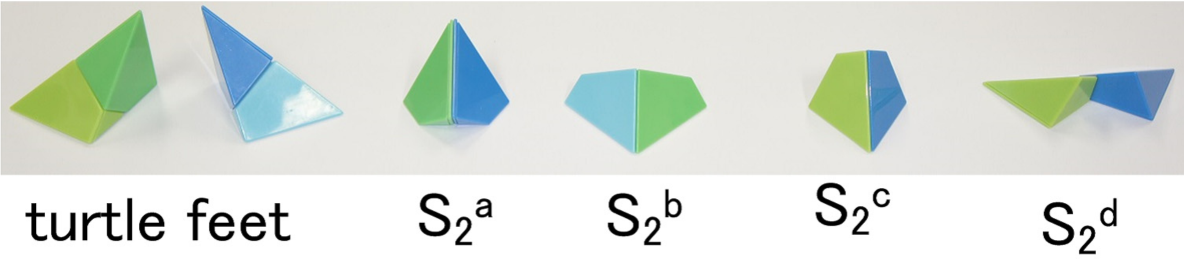

Fig. 11 All the possibilities of concatenations of 2 pentadra in pcs.

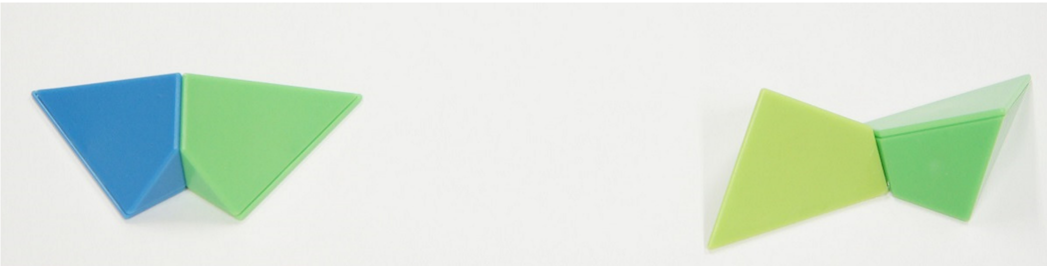

Fig. 12 Impossible concatenations of 2 pentadra in pcs for different sexes (left) and for the same sex (right).

Fig. 10 The tessellation T_P by pentadra.

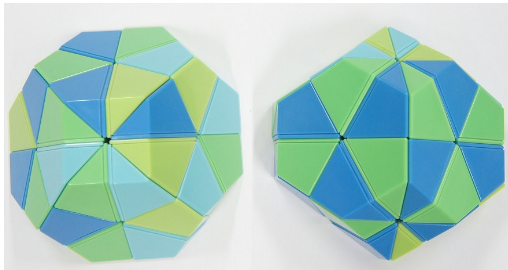

Fig. 13 Two kinds of half-TOs.

i.e., C is a turtle foot complex.

Proof:

Let C be a pc including at least one turtle foot as a subcomplex.

Suppose that C is not a turtle foot complex. Then there exists a turtle foot T which is a subcomplex of C and is adjacent with a non-coupled pentadron. Thus C is not convex and this is a contradiction. □

3.2 Characterizations of Solo Complices

If a pentadral complex C includes no coupled pairs, C is called a **solo** complex.

Let **TO** be the truncated octahedron which is a pentadral complex p_3 of 48 pentadra as in Fig. 4.

Theorem 4 A solo complex can never be extended further than the truncated octahedron TO, i.e., any solo complex is a subcomplex of TO.

Proof:

Consider the space tessellation by TOs in a face-to-face gluing manner (See Fig. 3). Any adjacent TOs sharing a hexagonal face have coupled complices on their intersection.

Suppose that a solo complex includes pentadra included in different TOs. Then the complex should include a pair of pentadra sharing a part of a hexagonal face of some TO in order for the complex to be convex. This results in a coupled pair, which is a contradiction. □

We call the minimal pc including given combination of pentadra in T_P a **pc convex hull** of the combination of pentadra.

Note that TO itself is a solo pentadral complex which fills the space. Any proper subcomplex of TO should be a subcomplex of one of the two **half-TO**s as in **Fig. 13** by the convexity.

Both of the half-TOs are space-fillers. Furthermore, any proper subcomplex of a half-TO is a subcomplex of the other half-TO. Thus we need to consider proper subcomplices of only one kind of half-TO. One can list up every translational, rotational, and reflectional equivalent classes of them by using pc convex hull and appropriate symmetry, although it is enough to list up solo complices with 1, 2, or 3 pentadra for our purpose. In fact, we have the complete list for the cases of solo complices with 1 or 2 pentadra so far.

Proposition 2 Every solo pentadral complex consisting of 3 pentadra is either S_3^N or S_3^T (**Fig. 14**).

S_3^N does not fill the space whereas S_3^T is a space-filling pc. Thus S_3^N is the non-space-filling pentadral complex with minimum number of pentadra, which is unique up to mirror reflection.

3.3 Seeds of a Space-filler and Pcs

Pentadron is not a seed since the solo complex S_3^N is not a space-filler. If we add partners for each of the three pentadra to S_3^N, we have a coupled complex consisting of 6 pentadra, denoted by M_6^N (See Fig. 16). The following proposition follows directly

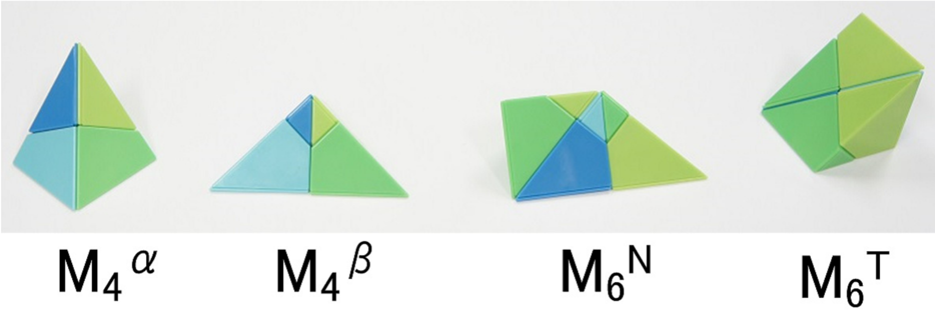

Fig. 16 Completions M_4^{α}, M_4^{β}, M_6^N and M_6^T.

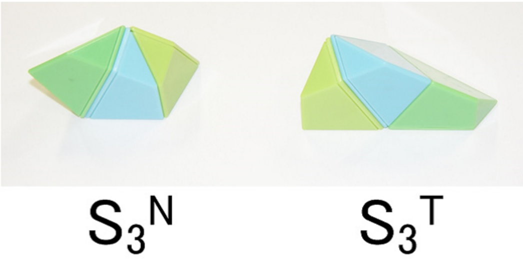

Fig. 14 The list of all the solo complices with 3 pentadra.

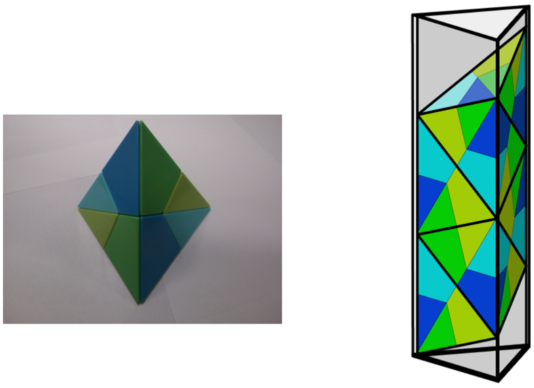

Fig. 15 Tetrapaks packed in the equilateral triangular prism.

from the definition of seed and the fact that M_6^N is also a not space-filler.

Proposition 3 If M_6^N is a P-complex then P is not a seed.

Especially, turtle foot, which is a space-filling tetrahedron in Fig. 2, is not a seed.

Any coupled complex is a **completion** of (not necessarily one) solo complices, i.e., they can be attained by adding partners for each pentadra to solo complices. Two solo pcs with the same completion are said to **complement** each other.

Tetrapak, also known as **Sommerville tetrahedron** (Ref. [8], Fig. 7), is a tetrahedron which can be made by 8 pentadra as in **Fig. 15**.

Conjecture 1 A tetrapak is a seed.

Note that any coupled complex with 2, 4, or 6 pentadra are not seeds, which can be confirmed by complementing the list of solo complices with 1, 2, or 3 pentadra. Couple pcs with 2 pentadra are completion of a pentadron, which is just a turtle foot. The pair S_2^a and S_2^b and the pair S_2^c and S_2^d have the same completion M_4^{α} and M_4^{β}, respectively; pcs in each pair complement each other. Solo pcs S_3^N and S_3^T have completions M_6^N and M_6^T, respectively (**Fig. 16**).

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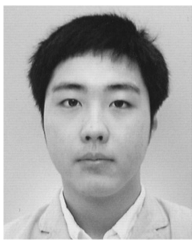

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