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Number of Three-point Tilings with Triangle Tiles

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Abstract: Three-point tiling is the problem to cover all the lattice points in a triangular region of the triangular lattice with triangle tiles that connect three adjacent lattice points. All the lattice points must be used by exactly one triangle tile. In this paper, we enumerate all the solutions and rotation symmetric solutions using ordered binary decision diagrams. In addition, the number of essentially different solutions, any two of which do not become identical by rotating and turning over, is computed.

Keywords: tiling, ordered binary decision diagram, enumeration

1. Introduction

Tiling is the problem to tile the plane or a finite region by a finite kinds of tiles [1], [2]. It is one of the most basic problems in combinatorial theory. Three-point tiling [3], [4], [5] is the problem to cover all the lattice points of the triangular lattice in a triangular region with tiles that connect three adjacent lattice points. In a tiling, all the points must be used by exactly one tile. This problem can also be considered as a problem to tile the hexagonal regular lattice by tiles which consist of three hexagons. Among the tiles that connect three adjacent points, we consider the triangle tile that covers a unit triangle.

Let the number of lattice points on an edge of the region be n. An example of a three-point tiling with triangle tiles for $n = 11$ is shown in **Fig. 1**. It is shown in Ref. [4] that the three-point tiling with triangle tiles is possible if and only if $n \pmod{12}$ is either $0, 2, 9$, or 11. However, the number of solutions for a region with n points on an edge is not known.

In this paper, we compute the number of solutions for the problem. To count the number of solutions, we use Ordered Binary Decision Diagrams (OBDDs) [6], [7]. An OBDD is a graph representation of a Boolean function. OBDDs are widely used in many applications due to their good properties [8]. Especially, OBDDs are very useful for enumerating all the solutions of combinatorial problems. To solve a combinatorial problem using OBDDs, we construct an OBDD that represents the restrictions to be a solution for the problem. Therefore, we can obtain the OBDD that represents all the solutions of the problem. Even though there are many solutions, they can often be represented by a compact OBDD because many of the solutions usually have partially the same structure. In addition, it is possible to extract the solutions with some property by operations on an OBDD.

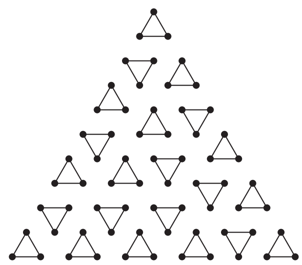

Fig. 1 An example of tiling.

Some solutions of the three-point tiling problem may be essentially the same. Two solutions are essentially the same if a solution becomes identical to the other solution by rotating and turning over the solution. In this paper, we enumerate all the solutions and the rotate symmetric solutions using OBDDs. Also, we have computed the number of essentially different solutions for $n \leq 33$.

2. Preliminaries

2.1 Three Point Tiling of the Triangular Lattice

A plane can be completely covered by equilateral triangles of the same size with a regular layout as shown in **Fig. 2** (a). A triangular lattice is the planar layout of triangles. The points in the plane where the corners of triangles are placed at are called lattice points. That is, the lattice points are the points represented by the linear combination $ie_1 + je_2$ of two unit vectors e_1 and e_2 in Fig. 2 (a) with integers i and j. The lattice point $ie_1 + je_2$ is represented by two-dimensional coordinates (i, j). Using the coordinates, unit triangles are the triangles that connect points $(i, j), (i, j+1)$ and $(i+1, j)$, or those that connect points $(i, j+1), (i+1, j)$ and $(i+1, j+1)$. The former ones are called *upward* and the latter ones are called *downward*.

In this paper, we consider the triangular region of the lattice. A triangular region is a part of the triangular lattice that consists of lattice points (i, j) satisfying $i \geq 0, j \geq 0$ and $i + j \leq n - 1$ for some integer n. Here, n is the number of lattice points on an edge

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of the triangular region. Figure 2 (b) is the triangular region with $n = 9$.

A three point tiling of a region of the triangular lattice with triangular tiles is a placement of triangular tiles with the size of a unit triangle such that each corner of a tile is placed on a lattice point and exactly one corner of a tile is placed on each lattice point.

2.2 Ordered Binary Decision Diagrams

An OBDD is a directed acyclic graph that represents a Boolean function. It has one source node and two sink nodes called *constant nodes* that are labeled by Boolean values 0 and 1 respectively. The nodes that are not sinks are called *variable nodes*. A variable node is labeled by a variable and has two outgoing edges called a 0-edge and a 1-edge respectively.

On any path from the source to a constant node, variables appear according to a total order of variables. The total order of variables is called the *variable order*.

Given an assignment to all the variables, the value of the function is computed by traversing from the source to one of the constant nodes according to the values of the variables. At a variable node, if the variable labeled to the node has value 1 (0 resp.), leave the node along the 1-edge (0-edge resp.). The value of the function is 1 (0 resp.) if the constant node is labeled 1 (0 resp.).

A node whose 1-edge and 0-edge point to the same node is called a redundant node. Nodes that are labeled by the same variable and represent the same function are called equivalent nodes. An OBDD which has no equivalent nodes and no redundant nodes is called a reduced OBDD. In this paper, OBDDs are assumed to be reduced. A Boolean function is uniquely represented by a reduced OBDD if the variable order is fixed. **Figure 3** is an example of the OBDD that represents a Boolean function $f = \overline{x_1}\ \overline{x_2}\ \overline{x_3} + x_1 x_2 + x_2 x_3$. The variable order of the OBDD is $x_1 x_2 x_3$.

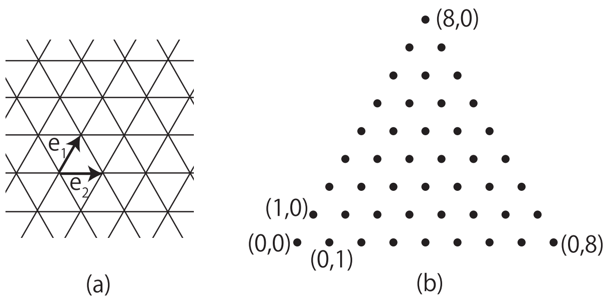


Fig. 2 Triangular grid and its triangular region.

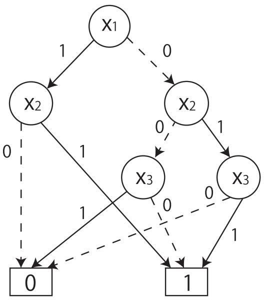


Fig. 3 An example of an OBDD.

3. Enumeration of Three-Point Tilings

3.1 Enumeration of All the Tilings

In this section, we propose the method to enumerate all the solutions of the three-point tiling problem with triangle tiles using OBDDs. The only input is a positive integer n, the number of lattice points on an edge of the triangular region.

Variables $t_{i,j}$ and $rt_{i,j}$ are defined as follows.

$$t_{i,j} = \begin{cases} 1 & \text{if there is a tile using points } (i, j), (i, j+1) \text{ and } (i+1, j), \\ 0 & \text{otherwise.} \end{cases}$$
$$(0 \le i, j \le n-2,\ 0 \le i+j \le n-2)$$

$$rt_{i,j} = \begin{cases} 1 & \text{if there is a tile using points } (i, j+1), (i+1, j) \\ & \text{and } (i+1, j+1), \\ 0 & \text{otherwise.} \end{cases}$$
$$(0 \le i, j \le n-3,\ 0 \le i+j \le n-3)$$

The triangles corresponding to variables $t_{i,j}$ are upward and those corresponding to $rt_{i,j}$ are downward as shown in **Fig. 4**.

The condition of the proper tiling is that each point is used by exactly one tile. That is, exactly one of the variables corresponding to the unit triangles around each point should be true. The condition for point (i, j) $(0 \le i, j \le n-1,\ 0 \le i+j \le n-1)$ is represented by the following function $P_{i,j}$.

$$\begin{aligned} P_{i,j} = {} & (t_{i,j-1} \wedge \overline{rt_{i,j-1}} \wedge \overline{t_{i,j}} \wedge \overline{rt_{i-1,j-1}} \wedge \overline{t_{i-1,j}} \wedge \overline{rt_{i-1,j}}) \\ & \vee (\overline{t_{i,j-1}} \wedge rt_{i,j-1} \wedge \overline{t_{i,j}} \wedge \overline{rt_{i-1,j-1}} \wedge \overline{t_{i-1,j}} \wedge \overline{rt_{i-1,j}}) \\ & \vee (\overline{t_{i,j-1}} \wedge \overline{rt_{i,j-1}} \wedge t_{i,j} \wedge \overline{rt_{i-1,j-1}} \wedge \overline{t_{i-1,j}} \wedge \overline{rt_{i-1,j}}) \\ & \vee (\overline{t_{i,j-1}} \wedge \overline{rt_{i,j-1}} \wedge \overline{t_{i,j}} \wedge rt_{i-1,j-1} \wedge \overline{t_{i-1,j}} \wedge \overline{rt_{i-1,j}}) \\ & \vee (\overline{t_{i,j-1}} \wedge \overline{rt_{i,j-1}} \wedge \overline{t_{i,j}} \wedge \overline{rt_{i-1,j-1}} \wedge t_{i-1,j} \wedge \overline{rt_{i-1,j}}) \\ & \vee (\overline{t_{i,j-1}} \wedge \overline{rt_{i,j-1}} \wedge \overline{t_{i,j}} \wedge \overline{rt_{i-1,j-1}} \wedge \overline{t_{i-1,j}} \wedge rt_{i-1,j}). \end{aligned}$$

Note that there are less than six unit triangles around the points that are on the edges of the triangular region. Thus, for such (i, j), $P_{i,j}$ includes the variables whose indices are not valid. We fix the variables with non-valid indices to false. As tiles must be placed on the triangles at the corners of the region, we can fix the values of $t_{0,0}$, $t_{0,n-2}$ and $t_{n-2,0}$ to be 1.

An assignment represents a three-point tiling if and only if the value of the following function F for the assignment is 1.

$$F = \bigwedge_{0 \le i, j \le n-1, 0 \le i+j \le n-1} P_{i,j}$$

The OBDD representing the Boolean function F is obtained by repeatedly executing logical operations on OBDDs, starting from the OBDDs representing the variables.

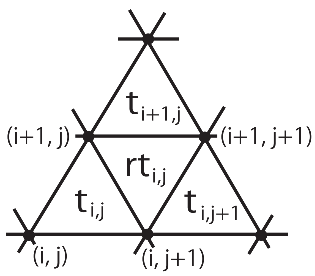
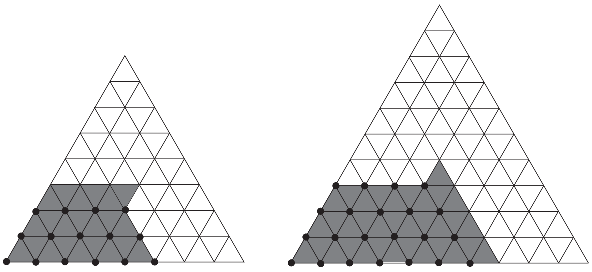


Fig. 4 Variables.



(a) Variables for n=9. (b) Variables for n=11.

Fig. 5 Variables for rotation symmetric tilings.

3.2 Enumeration of Rotation Symmetric Tilings

We call a tiling to be rotation symmetric if it is identical with the original tiling after rotation of 120° and 240°. In this section, we show how to enumerate the rotation symmetric tilings efficiently. On a rotation symmetric tiling, three variables corresponding to the triangles that overlap when rotated must have the same value. Thus we can decrease the number of variables to about one-third of the general case.

The triangles on which we have to assign variables depend on n. The variables to be used are as follows. When $n \pmod{12}$ is 2 or 11, the used variables are
$t_{(n-2)/3,(n-2)/3}$,
$t_{i,j}$ $(0 \le i \le (n-5)/3, 0 \le i+j \le n-3)$ and
$rt_{i,j}$ $(0 \le i \le (n-5)/3, 0 \le i+j \le n-4)$.
When $n \pmod{12}$ is 0 or 9, the used variables are
$rt_{(n-3)/3,(n-3)/3}$,
$t_{i,j}$ $(0 \le i \le (n-3)/3, 0 \le i+j \le n-3)$ and
$rt_{i,j}$ $(0 \le i \le (n-3)/3, 0 \le i+j \le n-4)$.
For example, when $n = 9$ and $n = 11$, we assign variables to the triangles in the gray area of **Fig. 5**.

The functions $P_{i,j}$ are similar to the general case. However, we have only to construct $P_{i,j}$ for the points that are in the middle or on the edges of the gray areas except the points on the right (top resp.) edge when $n \pmod{12}$ is 2 or 11 (0 or 9 resp.). That is, points (i, j) satisfying $0 \le i \le (n-2)/3$ and $0 \le i+j \le (2n-4)/3$ when $n \pmod{12}$ is 2 or 11, and those satisfying $0 \le i \le (n-3)/3$ and $0 \le i+j \le (2n-3)/3$ when $n \pmod{12}$ is 0 or 9, The points are shown by black dots in Fig. 5. When variables are not assigned to some of the unit triangles around a point, the variables that overlap with the triangles by rotation are used instead to compute $P_{i,j}$.

4. Number of Essentially Different Tilings

4.1 Duplicate Tilings

A tiling may become identical with another tiling by rotating it or by turning it over. We call a tiling which become identical with a given tiling, including the given tiling itself, is a duplicate of the tiling. The number of different duplicates depends on the tiling.

We divide the tilings into rotation symmetric ones and the other ones. For the tilings that are not rotation symmetric, there exist at most six duplicates as shown in **Fig. 6**. The left top tiling of the figure is the original tiling. The other tilings in the top row are obtained by rotating the original tiling. The tilings in the bottom row are obtained by turning over the above tiling.

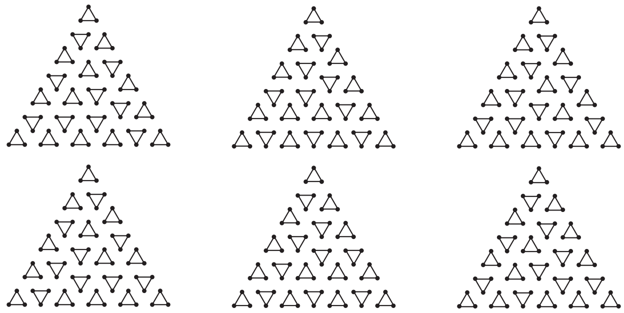


Fig. 6 Duplicate tilings.

Fig. 7 Tiling patterns near the corner.

Lemma 1 For any tiling that is not rotation symmetric, there exist six different duplicates.

Proof As the tiling is not rotation symmetric, any two of the three tilings obtained by rotation are not identical. By the same reason, any two of the three tilings obtained by turning them over are not identical. We will show that any of the three tilings obtained by rotation is not identical with any of the three tilings obtained by turning them over.

For any tiling, the pattern of tiles near the corners of the triangular region must be either of **Fig. 7**. The point at the top of the figure is the corner of the triangular region. Let the left one be pattern A and the right one be pattern B. Let a tiling which has three corners of pattern A be a AAA tiling and a tiling which has two corners of pattern A and one corner of pattern B be a AAB tiling. Similarly, BBB and ABB tilings can be defined. The tiling patterns on the corners do not change by rotation. For example, if a tiling is an AAA tiling, it remains to be an AAA tiling after rotation. We can observe that, by turning over a tiling, pattern A becomes pattern B and vice versa. Thus, after turning over a tiling, an AAA tiling becomes a BBB tiling and an AAB tiling becomes an ABB tiling. Therefore, no tiling can be identical with any of the three tilings obtained by turning it over. □

Lemma 2 For any rotation symmetric tiling, there exist only two different duplicates.

Proof As shown before, the tiling obtained by turning over a given tiling is not identical with the original one because they have different patterns on the corners. As a rotation symmetric tiling is identical with the ones obtained by rotation, there are only two duplicates that are not identical. □

4.2 Counting the Number of Essentially Different Tilings

In this section, we consider how to count the number of essentially different tilings.

Theorem 1 Let q, r, s be the number of all the AAA tilings, the number of rotation symmetric AAA tilings, and the number of AAB tilings which have pattern B at a fixed corner of the region, respectively. Then, the total number of essentially different tilings is $r + (q-r)/3 + s = \frac{1}{3}(q + 2r + 3s)$.

Proof First, we classify the tilings we have to count. As any

Table 1 Number of tilings.

n	N_{total}	N_{AAA}	N_{AAB}	N_{rot}	N_{diff}
2	1	–	–	–	1
9	2	1	0	1	1
11	8	1	1	1	2
12	12	3	1	0	2
14	72	9	9	0	12
21	185,328	23,634	23,010	0	30,888
23	4,736,520	587,676	593,528	0	789,420
24	21,617,456	2,722,666	2,695,354	112	3,602,984
26	912,370,744	113,597,576	114,195,932	488	152,062,116
33	3,688,972,842,502,560	461,440,189,850,352	461,015,410,466,976	59,808	614,828,807,123,632
35				433,136	
36				1,116,160	
38				5,913,328	
45				13,382,425,344	

BBB (ABB resp.) tiling is obtained by turning over an AAA (AAB resp.) tiling, we have only to count the number of AAA and AAB tilings. No AAB tilings can be rotation symmetric because the position of the corner with pattern B changes after rotation. Thus a rotation symmetric tiling must be an AAA tiling. Hence, the tilings we have to count can be classified into the following three cases.

- Rotation symmetric AAA tilings.
- AAA tilings that are not rotation symmetric.
- AAB tilings.

Now we consider the number of essentially different tilings for each case. First, all of the r rotation symmetric AAA tilings are essentially different. It is because non-identical rotation symmetric tilings never become identical after rotation. Next, we consider AAA tilings that are not rotation symmetric. As each tiling of this kind has three duplicate AAA tilings, the number of essentially different tilings is one-third of the total number of such tilings. The number of AAA tilings that are not rotation symmetric is $q-r$. Thus the number of essentially different tilings of this kind is $(q-r)/3$. Finally, similarly to the previous case, the number of essentially different AAB tilings is one-third of the total number of AAB tilings. As three AAB tilings obtained by rotating an AAB tiling have pattern B at different corners, the number of essentially different AAB tilings equals the number s of AAB tilings which have pattern B at a fixed position. In total, the total number of essentially different tilings is $r + (q-r)/3 + s$. □

The number of all the tilings can also be represented by q and s. As each AAA tiling has two duplicate tilings and each AAB tiling with pattern B at a fixed corner has six duplicate tilings, the number of all the tilings is $2q + 6s$.

5. Experimental Results

We have implemented programs to enumerate the following tilings using OBDDs.

- All the tilings.
- All the AAA tilings.
- All the AAB tilings which have pattern B at a fixed corner.
- All the rotation symmetric AAA tilings.

The number of the tilings are denoted by N_{total}, N_{AAA}, N_{AAB} and N_{rot} respectively. Also the number of essentially different tilings is denoted by N_{diff}.

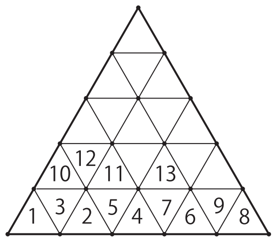
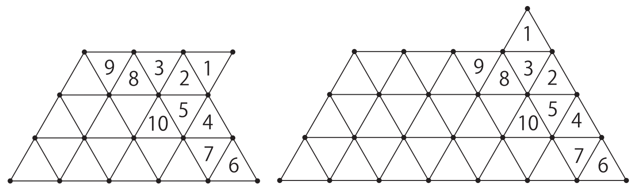


Fig. 8 Variable order.



(a) n (mod 12)=0,9 (b) n (mod 12)=2, 11

Fig. 9 Variable order for rotation symmetric tilings.

The variable orders we used are shown in **Fig. 8** and **Fig. 9**. Figure 9 shows the variable orders for enumerating rotation symmetric tilings. The variable orderings are the best ones among some orderings we have experimented. Note that values of some variables near the corner are fixed when we enumerate the tilings with fixed patterns of corners.

The experiments are executed on SUNW UltraSPARC-IIIi*2 (1.6 GHz) with 16 GB memory using CUDD package [9]. The number of tilings obtained by the enumeration programs are shown in **Table 1**. The rightmost column is the number of essentially different tilings computed from the results.

We could enumerate the rotation symmetric AAA tilings for $n \leq 45$ and other tilings for $n \leq 33$. The entries with − means that such tilings do not exist clearly. Empty entries mean that the OBDD size became too large to handle on the memory. Note that we can confirm that the number of all the tilings equals to $2q + 6s$ as claimed in the previous section.

We have also implemented enumeration programs using zero-suppressed BDDs (ZDDs) [10], which is a variation of OBDDs, and compared the efficiency with the implementation using OBDDs. The number of nodes representing all the solutions and the execution time for enumerating all the solutions are shown in **Ta-**

Table 2 Comparison of implementations using OBDDs and ZDDs.

	number of nodes		execution time(s)	
n	OBDD	ZDD	OBDD	ZDD
9	120	29	0.00	0.02
11	440	100	0.00	0.05
12	674	149	0.00	0.08
14	2,155	473	0.01	0.19
21	100,036	21,934	0.55	2.10
23	320,482	70,654	2.16	4.15
24	561,698	123,974	4.18	6.61
26	1,819,043	403,296	20.15	17.07
33	111,801,774	—	1548.03	—

ble 2. When we used ZDDs, the size of the ZDD became too large to handle for $n = 33$. Though the numbers of nodes of the obtained ZDDs are smaller than those of OBDDs, the peak number of nodes is larger on ZDDs.

6. Conclusions

In this paper, we enumerated three-point tilings with triangle tiles using OBDDs and computed the number of essentially different tilings for $n \leq 33$. Also we enumerated rotation symmetric tilings for $n \leq 45$.

Though we could count the number of essentially different tilings, we did not obtain the OBDDs that represent only the tilings. To obtain such OBDDs, we must be able to extract one tiling among three *AAA* tilings which become identical by rotation. It still remains as a challenging problem to represent the number of solutions as a function of n.

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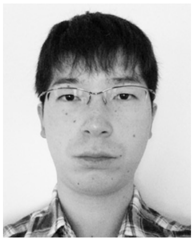

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