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On the Convergence Speed for a Class of Iterative Methods

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We derived two types of iterative methods, each containing two parameters and having cubic convergence for the zeros of all sufficiently regular functions. Our methods include Laguerre's, Ostrowski's, Halley's and Hansen and Patrick's methods. We established that our methods converge globally and monotonically to the real zeros of polynomials or certain entire functions. Further, when we find numerical solutions of the real zeros of the said functions by using the said methods, we established that as to the convergence speed, Ostrowski's method is the fastest, Halley's method is the slowest and our methods excepting the said two methods are intermediate. In this paper, we discuss the convergence speed in one of the said two types of our methods. Here, in the case where one of the two parameters contained in the said type of methods is given, we show how to derive the fastest method by the suitable choice of the other parameter.

1. Introduction

In order to find the numerical solutions of nonlinear scalar equations, many types of iterative methods have been derived. These methods have been treated in many books[1)-4)] and papers.[6)-10)] We will consider the computation of the numerical solutions of two types of nonlinear scalar equations.

The one type of the said equations is given by the following form:

$$f(x)\equiv\prod_{k=1}^{r}(x-\zeta_k)=0 \qquad (1.1)$$

where $r>1$ and $\zeta_{k+1}\geqq\zeta_k$ $(k=1, \cdots, r-1)$.

The other is given by the following form:

$$f(x)\equiv x^p\exp(a+bx-cx^2)\prod_{k=i}^{\infty}\left(1-\frac{x}{\alpha_k}\right)\times e^{x/\alpha_k}=0 \qquad (1.2)$$

where p is a non-negative integer; a, b, and c are real with $c\geqq 0$; and all α_k are real with $\sum\alpha_k^{-2}<\infty$.

Then it has been shown that Ostrowski's method (Ref. 2), pp. 110-115), Laguerre's method (Ref. 2), pp. 353-362), Hansen and Patrick's methods,[6)] and two types of our methods[5),10),11)] converge globally and monotonically to the zeros of $f(x)$. Recently, a new theorem of global convergence for Halley's method was obtained by using the concept of the degree of logarithmic convexity.[12)] In the previous paper,[13)] we showed that as to the convergence speed, Ostrowski's method is the fastest, Halley's method is the slowest, and two types of our methods excepting the said two methods are intermediate under some assumptions.

Here, we will consider the iteration functions for $R(X)$ of Ex. 1 in (Ref. 10), p. 188).

By putting $\theta+\frac{1}{2}=\gamma$, the said iteration functions can be expressed in the form:

$$\Phi_\beta(x)=x-hR_\beta(X) \qquad (1.3)$$

where $h\equiv h(x)=\frac{f(x)}{f'(x)}$, $X\equiv X(x)=h\times\frac{f''(x)}{f'(x)}$, $R_\beta(X)=\frac{\gamma X+1}{\beta X^2+\left(\gamma-\frac{1}{2}\right)X+1}$

and both β and γ are parameters.

In addition, β and γ satisfy the following inequality:

$$\beta\geqq\frac{1}{2}\gamma^2 \quad (0\geqq\gamma\geqq-1) \qquad (1.4)$$

In section 2, we will discuss the convergence speed in a class of iterative methods for $\Phi_\beta(x)$ in Eq. (1.3).

In section 3, we will derive some sufficient conditions for $\Phi_\beta(x)$ in Eq. (1.3) to be strictly increasing on a certin set.

In section 4, we will show some examples.

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2. Convergence Speed

We consider the iterative methods for Eq. (1.3).

Then, the said methods can be represented in the following form:

$$x_{n+1}=\Phi_\beta(x_n) \quad (n=0, 1, \cdots) \qquad (2.1)$$

where $\Phi_\beta(x)=x-hR_\beta(X)$, and $R_\beta(X)$ $=\dfrac{\gamma X+1}{\beta X^2+\left(\gamma-\frac{1}{2}\right)X+1}$.

Let $f(x)$ be given by the form in Eq. (1.1). Then, for $\forall x\in E$, we can define the associated zero of $f(x)$, $\zeta(x)$ (See Ref. 10)). On the monotonic convergence, we have:

● **Theorem 1.** Let $f(x)$ be given by the form in Eq. (1.1).

Then, if $\beta\geqq\frac{1}{2}\gamma^2$ $(0\geqq\gamma\geqq-1)$ and if we choose the real starting value in (2.1) x_0 such that $f(x_0)f'(x_0)\neq 0$, then we have

$x_n\downarrow\zeta(x_0)$ $(h(x_0)>0)$ and $x_n\uparrow\zeta(x_0)$ $(h(x_0)<0)$ $(n=0, 1, 2, \cdots)$.

● **Proof.** In Ref. 10), it was shown that this theorem is valid for $-\frac{1}{2}\gamma\geqq\beta\geqq\frac{1}{2}\gamma^2$.

Let γ be fixed for $0\geqq\gamma\geqq-1$, and let X be fixed for $X<1$.

Then, $R_\beta(X)$ is a strictly decreasing function of β. Also, since $R_{-\frac{1}{2}\gamma}(X)=\dfrac{1}{1-\frac{1}{2}X}$, the following inequality is valid for $\beta>-\frac{1}{2}\gamma$ $(0\geqq\gamma\geqq-1)$:

$$\frac{1}{1-\frac{1}{2}X}\geqq\frac{\gamma X+1}{\beta X^2+\left(\gamma-\frac{1}{2}\right)X+1}>0 \quad (X<1).$$

Therefore, this theorem also is valid for $\beta>-\frac{1}{2}\gamma$ $(0\geqq\gamma\geqq-1)$.

Consequently, this theorem is valid for $\beta\geqq\frac{1}{2}\gamma^2$ $(0\geqq\gamma\geqq-1)$.

Next, on the convergence speed of the methods (2.1), we have:

● **Theorem 2.** Let $f(x)$ be given by the form in Eq. (1.1), let β_0 and β_1 be given by the inequality $\beta_0>\beta_1\geqq\frac{1}{2}\gamma^2$ $(0\geqq\gamma\geqq-1)$ and let the iterative methods with β_0 and β_1 be given by the following forms respectively:

$y_{n+1}=\Phi_{\beta_0}(y_n)$, $z_{n+1}=\Phi_{\beta_1}(z_n)$ $(n=0, 1, \cdots)$.

Next, if the iteration function $\Phi_{\beta_0}(x)$ or $\Phi_{\beta_1}(x)$ is strictly increasing on E, and if we choose the real starting values y_0 and z_0 respectively such that $y_0=z_0$ and $f(y_0)f'(y_0)f''(y_0)\neq 0$, then we have

$\zeta(y_0)<z_n<y_n$ $(h(y_0)>0)$ and $\zeta(y_0)>z_n>y_n$ $(h(y_0)<0)$ $(n=1, 2, \cdots)$.

● **Proof.** Since $\beta_0>\beta_1\geqq\frac{1}{2}\gamma^2$ $(0\geqq\gamma\geqq-1)$, we have

$$R_{\beta_1}(X)\geqq R_{\beta_0}(X)>0 \quad (X<1). \qquad (2.2)$$

By applying (2.2) and Theorem 1, we can prove this theorem in a way similar to that of Theorem 7 in Ref. 13).

● **Corollary 1.** Let β_0 be given by the inequality $\beta_0>-\frac{1}{2}\gamma$ $(0\geqq\gamma\geqq-1)$.

Then, Halley's method is faster than the iterative methods with β_0 under the same assumptions of Theorem 2 on the real starting values of these methods.

● **Proof.** By replacing β_1 in Theorem 2 by $-\frac{1}{2}\gamma$ and applying the fact that Halley's iteration function $\Phi_{-\frac{1}{2}\gamma}(x)$ is strictly increasing on E,[13] it can be shown that this corollary is valid.

Next, let $f(x)$ be given by the form in Eq. (1.2). Further, for $\forall x\in E$, we can define the associated zero of $f(x)$, $\alpha(x)$ (See Ref. 11)). Then, on the monotonic convergence and the convergence speed, we have:

● **Theorem 3.** Let $f(x)$ be given by the form in Eq. (1.2). Then, if $\beta\geqq\frac{1}{2}\gamma^2$ $(0\geqq\gamma\geqq-1)$ and if we choose the real starting values in (2.1) x_0 such that $f(x_0)f'(x_0)\neq 0$, and x_0 is neither less nor greater than all α_k, we have

$x_n\downarrow\alpha(x_0)$ $(h(x_0)>0)$ and $x_n\uparrow\alpha(x_0)$ $(h(x_0)<0)$ $(n=0, 1, 2, \cdots)$.

● **Theorem 4.** Let $f(x)$ be given by the form in Eq. (1.2), let β_0 and β_1 be given by the inequality $\beta_0>\beta_1\geqq\frac{1}{2}\gamma^2$ $(0\geqq\gamma\geqq-1)$ and let the iterative methods with β_0 and β_1 be given by the following forms respectively:

$y_{n+1}=\Phi_{\beta_0}(y_n)$, $z_{n+1}=\Phi_{\beta_1}(z_n)$ $(n=0, 1, \cdots)$.

Next, if the iteration function $\Phi_{\beta_0}(x)$ or

$\Phi_{\beta_1}(x)$ is strictly increasing on E, and if we choose the real starting values y_0 and z_0 respectively such that $y_0=z_0$, $f(y_0)f'(y_0)f''(y_0)\neq 0$ and y_0 is neither less nor greater than all α_k, we have

$$\alpha(y_0)<z_n<y_n\ (h(y_0)>0) \text{ and } \alpha(y_0)>z_n>y_n\ (h(y_0)>0)\quad (n=0, 1, 2, \cdots).$$

● **Corollary 2.** Let β_0 be given by the inequality $\beta_0>-\frac{1}{2}\gamma$ $(0\geqq\gamma\geqq-1)$.

Then, Halley's method is faster than the iterative methods with β_0 under the same assumptions of Theorem 4 on the real starting values of these methods.

These theorems and Corollary 2 can be shown in the way similar to the proof of Theorem 1, Theorem 2 and Corollary 1.

3. Sufficient Conditions for Monotonicity

Let $f(x)$ be given by the form in Eq. (1. 1) and let the set $\{x;\ f(x)f'(x)\neq 0$ and x is real$\}$ be denoted by E. Then, in section 2 of Ref. 13), for $\forall x\in E$, we derived the following relations:

$$1-X=h^2\sum_{k=1}^{r}\frac{1}{(x-\zeta_k)^2}>0 \qquad (3.1)$$

$$\frac{dh}{dx}=\frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right)=\frac{\{f'(x)\}^2-f(x)f''(x)}{\{f'(x)\}^2}=1-X \qquad (3.2)$$

$$h\frac{dX}{dx}=-2(1-X)^2+2h^3\sum_{k=1}^{r}\frac{1}{(x-\zeta_k)^3} \qquad (3.3)$$

$$\frac{h^3}{(1-X)^{3/2}}\sum_{k=1}^{r}\frac{1}{(x-\zeta_k)^3}<1. \qquad (3.4)$$

Next, putting $Q(X)=\gamma X+1$, and $P(X)=\beta X^2+\left(\gamma-\frac{1}{2}\right)X+1$, we can represent the iteration functions in Eq. (1. 3) $\Phi_\beta(x)$ by the following form:

$$\Phi_\beta(x)=x-\frac{Q(X)}{P(X)}h.$$

Here, β and γ satisfy the following inequality:

$$-\frac{1}{2}\gamma\geqq\beta\geqq\frac{1}{2}\gamma^2. \qquad (3.5)$$

Defferentiating $\Phi_\beta(x)$ with respect to x, we have

$$\Phi_\beta'(x)=1-\frac{Q'P-QP'}{P^2}h\frac{dX}{dx}-\frac{Q}{P}\frac{dh}{dx} \qquad (3.6)$$

where $P\equiv P(X)$, $Q\equiv Q(X)$, $P'\equiv\frac{dP}{dX}$, $Q'\equiv\frac{dQ}{dX}$.

Further, putting $F=Q'P-QP'$ and applying (3. 2) and (3. 3) to the right side in (3. 6), we have

$$\begin{aligned}\Phi_\beta'(x)&=1-\frac{F}{P^2}\left\{-2(1-X)^2+2h^3\sum_{k=1}^{r}\frac{1}{(x-\zeta_k)^3}\right\}-\frac{Q}{P}(1-X)\\&=\frac{1}{P^2}\left\{P^2+2F(1-X)^2-PQ(1-X)-2Fh^3\sum_{k=1}^{r}\frac{1}{(x-\zeta_k)^3}\right\}.\end{aligned} \qquad (3.7)$$

Since $F=-\gamma\beta X^2-2\beta X+\frac{1}{2}$, it follows from (3. 5) that $F'=-2\gamma\beta X-2\beta<-2\gamma\beta-2\beta=-2\beta(\gamma+1)\leqq 0$ $(X<1)$. Further, since $F(1)=-\gamma\beta-2\beta+\frac{1}{2}=-\beta(\gamma+2)+\frac{1}{2}\geqq\frac{1}{2}(\gamma+1)^2$, we have $F(X)>0$ $(X<1)$.

Consequently, it follows from (3. 4) and (3. 7) that if the following inequality is valid:

$$P^2+2F(1-X)^2-PQ(1-X)-2F(1-X)^{3/2}\geqq 0\quad (\forall x\in E) \qquad (3.8)$$

then $\Phi_\beta(x)$ is strictly increasing on E.

In the following, we derive some sufficient conditions for the inequality (3. 8) to be valid.

First of all, the left side in (3. 8)

$$\begin{aligned}&=P\{P-Q(1-X)\}+2F(1-X)^{3/2}\times(\sqrt{1-X}-1)\\&=\left\{\beta(\beta+\gamma)X^2+\gamma\left(\beta+\gamma-\frac{1}{2}\right)X+\beta+\frac{3}{2}\gamma-\frac{1}{4}\right\}X^2+\frac{1}{2}X+(-2\gamma\beta X^2-4\beta X+1)\\&\quad\times(1-X)^{3/2}(\sqrt{1-X}-1).\end{aligned}$$

Next, putting $\sqrt{1-X}=t$ and denoting the left side in (3. 8) $G(t)$, we have

$$\begin{aligned}G(t)&=\left\{\beta(\beta+\gamma)X^2+\gamma\left(\beta+\gamma-\frac{1}{2}\right)X+\beta+\frac{3}{2}\gamma-\frac{1}{4}\right\}(1-t^2)^2\\&\quad+\frac{1}{2}(1-t^2)+(-2\gamma\beta X-4\beta)\times(1-t^2)t^3(t-1)+t^3(t-1)\\&=(1-t)^2\left[\left\{\beta(\beta+\gamma)X^2+\gamma\left(\beta+\gamma-\frac{1}{2}\right)X+\beta+\frac{3}{2}\gamma-\frac{1}{4}\right\}(1+t)^2\right.\end{aligned}$$

$$+(2\gamma\beta X+4\beta)(1+t)t^3+t^2+t+\frac{1}{2}\Big]=(1-t)^2K(t)\quad(t>0)\tag{3.9}$$

where $K(t)$ represents the expression in the bracket of $G(t)$.

Since $X^2(1+t)^2=t^6+2t^5-t^4-4t^3-t^2+2t+1$, $X(1+t)^2=-t^4-2t^3+2t+1$, and $X(1+t)\times t^3=-t^6-t^5+t^4+t^3$, we have

$$K(t)=\sum_{m=0}^{6}K_mt^m\quad(t>0)\tag{3.10}$$

where

$$K_0=\left(\beta+\gamma+\frac{1}{2}\right)^2\geqq 0,$$

$$K_1=2\left(\beta+\gamma+\frac{1}{2}\right)^2\geqq 0,$$

$$K_2=\left(\frac{3}{2}-\beta\right)\left(\beta+\gamma+\frac{1}{2}\right)\geqq 0$$

$$\left(\because\ \frac{1}{2}\geqq\beta\geqq 0 \text{ and } \beta+\gamma+\frac{1}{2}\geqq\frac{1}{2}\gamma^2+\gamma+\frac{1}{2}=\frac{1}{2}(\gamma+1)^2\right),$$

$$K_3=-4\beta^2-4\beta\gamma+4\beta-2\gamma^2+\gamma,$$

$$K_4=-\beta^2+4\beta-\gamma^2+\frac{1}{2}\gamma,\ K_5=2\beta^2\geqq 0, \text{ and}$$

$$K_6=\beta(\beta-\gamma)\geqq 0.$$

Differentiating $K(t)$ in Eq. (3.10) again and again, we have

$$K'(t)=6K_6t^5+5K_5t^4+4K_4t^3+3K_3t^2+2K_2t+K_1,$$

$$K''(t)=30K_6t^4+20K_5t^3+12K_4t^2+6K_3t+2K_2,$$

$$K^{(3)}(t)=120K_6t^3+60K_5t^2+24K_4t+6K_3,$$

$$K^{(4)}(t)=360K_6t^2+120K_5t+24K_4, \text{ and}$$

$$K^{(5)}(t)=720K_6t+120K_5.$$

$\Phi_{-\frac{1}{2}\gamma}(x)$ coincides with Halley's iteration function which is strictly increasing on E.[13] Hereafter, from (3.5) we concentrate on the case of

$$-\frac{1}{2}\gamma>\beta\geqq\frac{1}{2}\gamma^2.$$

Since $K_4-\frac{1}{2}K_3=\beta(\beta+2\gamma+2)>0$, it follows that if $K_3\geqq 0$, then we have $K_4>0$. On the other hand, since $K_m\geqq 0$ $(m=0, 1, 2, 5, 6)$, we have $K(t)>0$ $(t>0)$. Hence, if $K_3\geqq 0$, then $\Phi_\beta(x)$ is strictly increasing on E.

Next, let us consider the case of $K_3<0$. Since $K^{(5)}(t)>0$, $K^{(3)}(t)$ is convex for $t>0$.

Then the graphs of the functions $y=K^{(i)}(t)$ $(t>0, i=1, 2, 3)$ and the signs of $K(t)$ $(t>0)$ are illustrated in **Fig. 1**.

In section 2 of Ref. 13), when $f(x)$ is given by the form in Eq. (1.2), for $\forall x\in E$, we derived the following relations:

$$1-X=h^2\left\{\frac{p}{x^2}+2c+\sum_{k=1}^{\infty}\frac{1}{(x-\alpha_k)^2}\right\}>0$$

$$h\frac{dX}{dx}=-2(1-X)^2+2h^3\times\left\{\frac{p}{x^3}+\sum_{k=1}^{\infty}\frac{1}{(x-\alpha_k)^3}\right\}$$

$$\frac{h^3}{(1-X)^{3/2}}\left\{\frac{P}{x^3}+\sum_{k=1}^{\infty}\frac{1}{(x-\alpha_k)^3}\right\}<1.$$

Therefore, also in the case where $f(x)$ is given by the form in Eq. (1.2), it can be shown that the said results are valid.

Finally, on the monotonicity of $\Phi_\beta(x)$, from

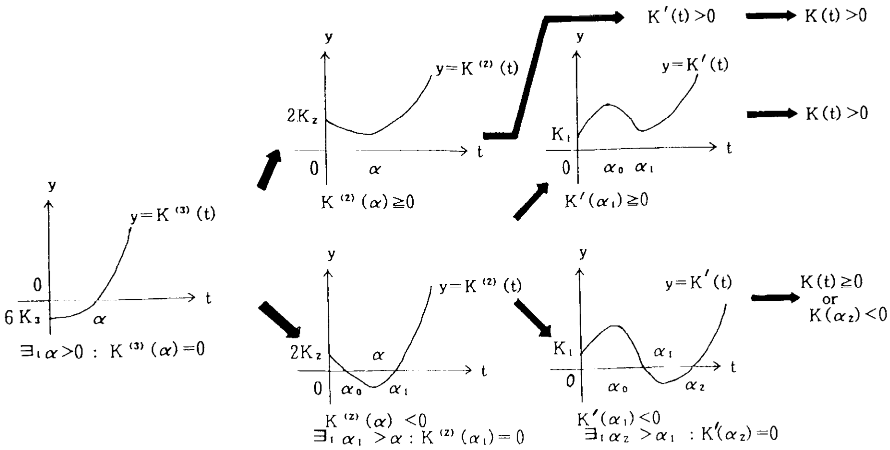

Fig. 1 The graphs of the functions $y=K^{(i)}(t)$ $(t>0, i=1, 2, 3)$ and the signs of $K(t)$.

the said results we have:

● **Theorem 5.** Let $f(x)$ be given by the form in Eq. (1. 1) or Eq. (1. 2), let γ and β be given by the inequality (3. 5) and let $K(t)$ be given by Eq. (3. 10). Then

[I] If $K_3 \geqq 0$, then $\Phi_\beta(x)$ is strictly increasing on E.

[II] If $K_3 < 0$, then $K^{(3)}(t) = 0$ $(t > 0)$ has the only one solution α.

(1) If $K^{(2)}(\alpha) \geqq 0$, then $\Phi_\beta(x)$ is strictly increasing on E.

(2) If $K^{(2)}(\alpha) < 0$, then $K^{(2)}(t) = 0$ $(t > 0)$ has the only one solution α_1 such that $\alpha_1 > \alpha$.

(3) If $K'(\alpha_1) \geqq 0$, then $\Phi_\beta(x)$ is strictly increasing on E.

(4) If $K'(\alpha_1) < 0$, then $K'(t) = 0$ $(t > 0)$ has the only one solution α_2 such that $\alpha_2 > \alpha_1$.

(5) If $K(\alpha_2) \geqq 0$, then $\Phi_\beta(x)$ is strictly increasing on E.

(6) If $K(\alpha_2) < 0$, then $\Phi_\beta(x)$ is not increasing on E.

Remark 1. Let γ be fixed for $0 \geqq \gamma \geqq -1$, let t be fixed for $t > 0$, and let β be given by the inequality (3. 5). Then, since

$$\frac{\partial K_0}{\partial \beta} = 2\left(\beta + \gamma + \frac{1}{2}\right) \geqq 0, \quad \frac{\partial K_1}{\partial \beta} = 2\frac{\partial K_0}{\partial \beta},$$

$$\frac{\partial K_2}{\partial \beta} = -2\beta - \gamma + 1 \geqq 1,$$

$$\frac{\partial K_3}{\partial \beta} = -8\beta - 4\gamma + 4 \geqq 4,$$

$$\frac{\partial K_4}{\partial \beta} = -2\beta + 4 \geqq 3,$$

$$\frac{\partial K_5}{\partial \beta} = 4\beta, \text{ and } \frac{\partial K_6}{\partial \beta} = 2\beta - \gamma \geqq 0,$$

$K(t)$ $(t > 0)$ is a strictly increasing function of β.

Therefore, in the case of Theorem 5[II] (6), by increasing the values of β gradually, it can also be seen that at last $\Phi_\beta(x)$ turns into a strictly increasing function on E.

4. Examples

1. Let γ be fixed for $-1 \leqq \gamma \leqq 0$, and let β be given by the inequality $\beta \geqq \frac{1}{2}\gamma^2$ $(0 \geqq \gamma \geqq -1)$. Then, in accordance with Theorem 5 and Remark 1, let us find the value of β as small as possible under the condition that $K(t)$ $(t > 0)$ is nonnegative. Next, let us give an example.

Let the closed interval $\{\gamma : 0 \geqq \gamma \geqq -1\}$ be divided into 2^8 equal segments by the points of division

$$\gamma_k = -\left(\frac{1}{2}\right)^8 k \quad (k = 0, 1, \cdots, 2^8)$$

and let β, which corresponds to γ_k be given by

$$\beta_{k,l} = \frac{1}{2}\gamma_k^2 + \left(\frac{1}{2}\right)^{13} l \quad (l = 0, 1, \cdots).$$

At first, finding γ_k and $\beta_{k,0} = \frac{1}{2}\gamma_k^2$ for given k, we investigate the sign of $K(t)$ in accordance with Theorem 5. Then if $K(t) \geqq 0$, then $\beta_{k,0}$ is the desired β. If $K(t) < 0$, replacing $\beta_{k,0}$ by $\beta_{k,1}$, we investigate the sign of $K(t)$ in accordance with Theorem 5. By repetition of the same process, it follows from Remark 1 that finally the sign of $K(t)$ becomes nonnegative. If $K(t)$ first becomes nonnegative for some $\beta_{k,l}$, then this $\beta_{k,l}$ is the desired β. Thus we can find the desired β for each k $(1 \leqq k \leqq 2^8 - 1)$ one after another. The suffixes k and l of the desired $\beta_{k,l}$ are tabulated in **Table 1**.

As to the convergence speed, if γ_k is given, then it follows from Table 1, Theorem 2, and Theorem 4 that the iterative methods with the desired $\beta_{k,l}$ are faster than those with β such that $\beta > \beta_{k,l}$, and slower than that with $\beta_{k,0} = \frac{1}{2}\gamma_k^2$ $(k = 1\ (1)\ 2^7 - 1, 2^7 + 1\ (1)\ 2^8 - 1)$. We computed the numerical solutions of α, α_1 and α_2 in Theorem

Table 1 The suffixes k and l of the desired $\beta_{k,l}$.

k	l	k	l	k	l
1	2	49-52	20	129-144	1
2	3	53-55	19	145-151	2
3	4	56-58	18	152-157	3
4	5	59-61	17	158-162	4
5	7	62-64	16	163-167	5
6	8	65-67	15	168-172	6
7	9	68-70	14	173-176	7
8	10	71-73	13	177-182	8
9	11	74,75	12	183-187	9
10,11	12	76-78	11	188-194	10
12	13	79-81	10	195-220	11
13	14	82-84	9	221-225	10
14,15	15	85-87	8	226-229	9
16	16	88-90	7	230-233	8
17,18	17	91-93	6	234-236	7
19,20	18	94-97	5	237-239	6
21,22	19	98-101	4	240,241	5
23-25	20	102-106	3	242-244	4
26-30	21	107-112	2	245-247	3
31-42	22	113-127	1	248-250	2
43-48	21	128	0	251-255	1

5 by using the bisection method. All calculations are made with quadruple precision.

2. If $|X|\ll 1$, then we have

$$\frac{1}{\sqrt{1-X}}-R_{\frac{1}{2}\gamma^2}(X)$$
$$=\frac{1}{8}(2\gamma+1)^2X^2+\frac{1}{16}(2\gamma+1)$$
$$\times(-4\gamma^2+2\gamma+3)X^3$$
$$+\left[\frac{35}{128}-\left\{\frac{1}{4}\gamma^4-\frac{3}{2}\left(\gamma-\frac{1}{2}\right)^2\gamma^2\right.\right.$$
$$+\left(\gamma-\frac{1}{2}\right)^4$$
$$\left.\left.+\gamma^3\left(\gamma-\frac{1}{2}\right)-\left(\gamma-\frac{1}{2}\right)^3\gamma\right\}\right]X^4$$
$$+O(X^5).$$

Hence

$$\frac{1}{\sqrt{1-X}}-R_{\frac{1}{2}\gamma^2}(X)$$
$$=\begin{cases}\frac{1}{128}X^4+O(X^5) & \left(\gamma=-\frac{1}{2}\right)\\ O(X^2) & \left(\gamma\neq-\frac{1}{2}\right).\end{cases}$$

On the other hand, if $\gamma=-\frac{1}{2}$ and $\beta=\frac{1}{8}$, then we have

$$K(t)=\frac{1}{64}(5t^6+2t^5-t^4-20t^3+11t^2$$
$$+2t+1)$$
$$=\frac{1}{64}(t-1)^2(5t^4+12t^3+18t^2$$
$$+4t+1)\geqq 0\quad(t>0).$$

Consequently, as to the convergence speed, if $|X|\ll 1$, then it follows from Theorem 2 and Theorem 4 that for the methods in (2.1), the method with $\gamma=-\frac{1}{2}$ and $\beta=\frac{1}{8}$ is the fastest.

3. We give the numerical examples of the convergence speed below. Let us call the methods in (2.1) the methods of the type (γ, β). Then, Halley's method is of the type $\left(\gamma, -\frac{1}{2}\gamma\right)$. Put $\beta=\beta_2=\frac{1}{2}\gamma^2$, $\beta=\beta_1=$ the desired β in Table 1, and $\beta=\beta_0>-\frac{1}{2}\gamma$.

Here, we treat the following types as the type (γ, β):

$$\left(-\frac{1}{4}, \beta_2\right)=\left(-\frac{1}{4}, \frac{1}{32}\right),$$
$$\left(-\frac{1}{4}, \beta_1\right)=\left(-\frac{1}{4}, \frac{1}{32}+\left(\frac{1}{2}\right)^9\right),$$
$$\left(-\frac{1}{4}, \beta_0\right)=\left(-\frac{1}{4}, \frac{1}{4}\right),$$
$$\left(-\frac{1}{2}, \beta_2\right)=\left(-\frac{1}{2}, \frac{1}{8}\right),$$
$$\left(-\frac{3}{4}, \beta_2\right)=\left(-\frac{3}{4}, \frac{9}{32}\right),$$
$$\left(-\frac{3}{4}, \beta_1\right)=\left(-\frac{3}{4}, \frac{9}{32}+5\cdot\left(\frac{1}{2}\right)^{12}\right),$$
$$\left(-\frac{3}{4}, \beta_0\right)=\left(-\frac{3}{4}, \frac{1}{2}\right), \text{ and } \left(\gamma, \frac{1}{2}\gamma\right).$$

We consider the following equation:

$$f(x)\equiv x^9+16x^8-93x^7-1586x^6-789x^5$$
$$+8244x^4-78247x^3-139594x^2$$
$$+494928x-282880=0$$

All the roots of this equation are

$$\zeta=-16,\ -8,\ 1,\ 1,\ 10,\ -4\pm i,\ 2\pm 3i.$$

Table 2 The values of d_n ($x_0=32.0$, $\zeta=10.0$).

Methods of the type (γ,β)	$\left(-\frac{1}{4},\beta_2\right)$	$\left(-\frac{1}{4},\beta_1\right)$	$\left(-\frac{1}{4},\beta_0\right)$	$\left(-\frac{1}{2},\beta_2\right)$	$\left(-\frac{3}{4},\beta_2\right)$	$\left(-\frac{3}{4},\beta_1\right)$	$\left(-\frac{3}{4},\beta_0\right)$	$\left(\gamma,-\frac{1}{2}\gamma\right)$	Ostrowski's method
d_1	1.4×10^{1}	1.4×10^{1}	1.7×10^{1}	1.3×10^{1}	1.1×10^{1}	1.1×10^{1}	1.8×10^{1}	1.6×10^{1}	1.1×10^{1}
d_2	8.5×10^{0}	8.5×10^{0}	1.2×10^{1}	6.2×10^{0}	4.7×10^{0}	4.8×10^{0}	1.4×10^{1}	1.0×10^{1}	4.7×10^{0}
d_3	4.2×10^{0}	4.3×10^{0}	8.7×10^{0}	2.1×10^{0}	1.1×10^{0}	1.1×10^{0}	1.1×10^{1}	6.5×10^{0}	1.1×10^{0}
d_4	1.5×10^{0}	1.6×10^{0}	5.8×10^{0}	2.2×10^{-1}	4.0×10^{-2}	4.8×10^{-2}	8.2×10^{0}	3.5×10^{0}	2.9×10^{-2}
d_5	1.8×10^{-1}	2.0×10^{-1}	3.4×10^{0}	3.4×10^{-4}	5.2×10^{-6}	9.2×10^{-6}	5.8×10^{0}	1.4×10^{0}	7.0×10^{-7}
d_6	5.7×10^{-4}	7.3×10^{-4}	1.7×10^{0}	1.1×10^{-12}	1.3×10^{-17}	6.9×10^{-17}	4.0×10^{0}	2.4×10^{-1}	1.0×10^{-20}
d_7	1.6×10^{-11}	3.5×10^{-11}	5.1×10^{-1}	0.0×10^{-31}	0.0×10^{-31}	0.0×10^{-31}	2.3×10^{0}	3.0×10^{-3}	0.0×10^{-31}
d_8	0.0×10^{-31}	0.0×10^{-31}	3.8×10^{-2}	……	……	……	1.0×10^{0}	7.2×10^{-9}	……
d_9	……	……	2.7×10^{-5}	……	……	……	2.2×10^{-1}	9.9×10^{-26}	……
d_{10}	……	……	9.3×10^{-15}	……	……	……	4.5×10^{-3}	0.0×10^{-31}	……
d_{11}	……	……	0.0×10^{-31}	……	……	……	4.3×10^{-8}	……	……
d_{12}	……	……	……	……	……	……	4.0×10^{-23}	……	……
d_{13}	……	……	……	……	……	……	0.0×10^{-31}	……	……

Next put

$$d_n = x_n - \zeta \quad (n=1, 2, \cdots).$$

The numerical examples of the convergence speed in nine methods are tabulated in **Table 2**.

The computational results in Table 2 substantiate the discussion presented above. Several other algebraic equations were tested and the results were consistent with those obtained above. All calculations are made with quadruple precision.

5. Concluding Remarks

Let $f(x)$ be given by the form in Eq. (1. 1) or Eq. (1. 2). Then it was shown that as to the convergence speed, for the methods in (2. 1) with $\beta \geqq -\frac{1}{2}\gamma$ $(-1 \leqq \gamma \leqq 0)$, Halley's method is the fastest and the slowest for the methods in (2. 1) with $-\frac{1}{2}\gamma \geqq \beta \geqq \frac{1}{2}\gamma^2$. Further, from the discussion in section 4, for the methods in (2. 1), the method with some β such that $\frac{1}{2}\gamma_k^2 \leqq \beta <$ the desired $\beta_{k,l}\left(\gamma_k \neq -\frac{1}{2}\right)$ is the fastest.

Hence it follows from Table 1 that when γ_k is given, as to the convergence speed, for the methods in (2. 1), the method with $\beta = \frac{1}{2}\gamma_k^2$ is nearly the fastest.

Roughly speaking, it can be concluded that when γ is given by $-1 \leqq \gamma \leqq 0$, as to the convergence speed, for the methods in (2. 1) with $\beta \geqq \frac{1}{2}\gamma^2$ $(-1 \leqq \gamma \leqq 0)$, the method with $\beta = \frac{1}{2}\gamma^2$ is nearly the fastest.

Let us define the function $\lambda_\gamma(X)$ as follows:

$$\lambda_\gamma(X) = \frac{\gamma X + 1}{\frac{1}{2}\gamma^2 X^2 + \left(\gamma - \frac{1}{2}\right)X + 1}$$

$$(-1 \leqq \gamma \leqq 0,\ X < 1).$$

Then we have

$$\lambda_\gamma(X) - \lambda_{\gamma_0}(X) = \frac{X^2(\gamma_0 - \gamma)A(X)}{2B_\gamma(X)B_{\gamma_0}(X)}$$

where $A(X) = \gamma\gamma_0 X + \gamma + \gamma_0 + 1$ and

$$B_\gamma(X) = \frac{1}{2}\gamma^2 X^2 + \left(\gamma - \frac{1}{2}\right)X + 1.$$

Put γ as follows:

$$\gamma = \begin{cases} -\frac{1}{2}(\gamma_0 + 1) & \text{for } \forall \gamma_0 \in \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, 0\right) \\ -\frac{1}{2} & \text{for } \gamma_0 = -\frac{1}{3}. \end{cases}$$

Since $A(X) = 0$ for $X = -\frac{\gamma + \gamma_0 + 1}{\gamma\gamma_0} (<0)$ and $B_\gamma(X)B_{\gamma_0}(X) > 0$, $\lambda_\gamma(X) - \lambda_{\gamma_0}(X)$ changes sign in a neighborhood of the point $X = -\frac{\gamma + \gamma_0 + 1}{\gamma\gamma_0}$.

Hence, there is not any constant γ_0 such that the following inequality holds for $\forall \gamma \in (-1, \gamma_0) \cup (\gamma_0, 0)$ and $\forall X < 1$:

$$\lambda_\gamma(X) \leqq \lambda_{\gamma_0}(X).$$

Finally, we see that there is not any constant γ_0 such that in the methods in (2. 1) with $\beta = \frac{1}{2}\gamma^2$ $(-1 < \gamma < 0)$, the method for $\gamma = \gamma_0$ is nearly the fastest.

Next, let us consider the case where Eq. (1. 1) and Eq. (1. 2) have complex zeros. Let ζ be any real zero of Eq. (1. 1), and $\bar{\zeta}_j$ also be a zero of Eq. (1. 1) if ζ_j is any complex zero of Eq. (1. 1). Then, if

$$h^2 \sum_{k=1}^{r} \frac{1}{(x - \zeta_k)^2} - \frac{h^2}{(x - \zeta)^2} > 0 \quad (\forall x \in E)$$

and

$$\sum_{k=1}^{r} \frac{1}{(x - \zeta_k)^3} \Big/ \left\{\sum_{k=1}^{r} \frac{1}{(x - \zeta_k)^2}\right\}^{3/2} < 1$$

$$(\forall x \in E),$$

we can draw the same conclusion as above. Further, let α be any real zero of Eq. (1. 2), and $\bar{\alpha}_j$ also be a zero of Eq. (1. 2) if α_j is any complex zero of Eq. (1. 2). Then, if

$$h^2\left\{\frac{p}{x^2} + 2c + \sum_{k=1}^{\infty} \frac{1}{(x - \alpha_k)^2}\right\} - \frac{h^2}{(x - \alpha)^2}$$

$$> 0 \quad (\forall x \in E)$$

and

$$\left\{p + x^3 \sum_{k=1}^{\infty} \frac{1}{(x - \alpha_k)^3}\right\} \Big/ \left\{p + 2cx^2 + x^2 \sum_{k=1}^{\infty} \frac{1}{(x - \alpha_k)^2}\right\}^{3/2} < 1$$

$$(\forall x \in E)$$

we can draw the same conclusion as above.

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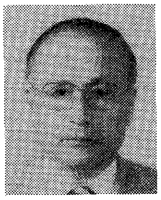

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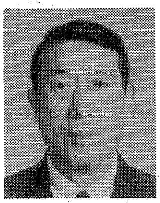

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