Technical Note

Quartic C^2 Interpolating Curve with Given Tangential Directions

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This paper presents a quartic C^2 curve passing through data points in given directions, useful in the field of computer graphics and computer aided geometric design. Tangential lengths are determined directly by a linear equation system without variational procedure. In the case of specifying the same tangential directions as the cubic C^2 interpolating curve, both curves are visually identical. By using these directions as defaults, the desired curves can easily be obtained.

1. Introduction

This paper presents a scheme to specify tangential directions at data points of a quartic C^2 interpolating curve. The curve with these human familiar parameters is useful in the field of computer graphics and computer aided geometric design.

Although cubic G^2 interpolating curves with given tangential directions were developed, these are not always practical because the control range of tangential directions is quite limited[1] or a complicated procedure is required[2]. The causes are from poor freedom of the piecewise cubic polynomial. On the contrary, the piecewise quartic one has excess freedom for these curves.

To control excess freedom of the quartic C^2 curve, our scheme minimizes the integral of square of its second derivative. The curve is almost the same as the cubic C^2 interpolating curve when it has the same tangential directions. By using these directions as defaults, we can obtain the desired curves in respect to satisfying tangential directions only at necessary points.

2. Scheme

We use the quartic B-splines with uniformly spaced knots of multiplicity 2 and an additional control point in each span (the S-splines[3]). The notations are as follows.

- given data points: $\boldsymbol{p}_1, \cdots, \boldsymbol{p}_i, \cdots, \boldsymbol{p}_n$,
- given unit tangent vectors: $\boldsymbol{t}_1, \cdots, \boldsymbol{t}_i, \cdots, \boldsymbol{t}_n$,
- unknown lengths of tangents: $x_1, \cdots, x_i, \cdots, x_n$,
- B-spline control points: $\boldsymbol{d}_1, \cdots, \boldsymbol{d}_{2i}, \cdots, \boldsymbol{d}_{2n+1}$.

The data point $\boldsymbol{p}_i$ corresponds to the control point $\boldsymbol{d}_{2i}$. Each span of the curve is expressed as follows.

$$\boldsymbol{r}_i(t)=\sum_{j=0}^{4} f_j(t)\,\boldsymbol{d}_{2i-1+j}, \quad 0\le t\le 1, \tag{1}$$

$$\begin{cases} f_0(t)=\dfrac{(1-t)^4}{4}, \quad f_1(t)=\dfrac{(1-t)^3(1+3t)}{2}, \\ f_2(t)=\dfrac{1}{4}+t+\dfrac{3t^2}{2}-5t^3+\dfrac{5t^4}{2}, \\ f_3(t)=\dfrac{(4-3t)\,t^3}{2}, \quad f_4(t)=\dfrac{t^4}{4}. \end{cases} \tag{2}$$

We have two relations in each data point.

$$\boldsymbol{d}_{2i-1}+2\boldsymbol{d}_{2i}+\boldsymbol{d}_{2i+1}=4\boldsymbol{p}_i, \tag{3}$$

$$-\boldsymbol{d}_{2i-1}+\boldsymbol{d}_{2i+1}=x_i\boldsymbol{t}_i, \tag{4}$$

Therefore, control points $\{\boldsymbol{d}_i\}_{i=1}^{2n}$ are expressed with unknown $\{x_i\}_{i=1}^{n}$ and $\boldsymbol{d}_{2n+1}$.

$$\boldsymbol{d}_{2i-1}=-\sum_{j=i}^{n} x_i\boldsymbol{t}_j+\boldsymbol{d}_{2n+1} \tag{5}$$

$$\boldsymbol{d}_{2i}=2\boldsymbol{p}_i+\frac{x_i\boldsymbol{t}_i}{2}+\sum_{\substack{j=i+1\\ j<n}}^{n} x_j\boldsymbol{t}_j-\boldsymbol{d}_{2n+1} \tag{6}$$

$$\boldsymbol{d}_{2n+1}=\begin{bmatrix} x_{n+1}\\ x_{n+2}\end{bmatrix}. \tag{7}$$

Unknowns $\{x_i\}_{i=1}^{n+2}$ are solved with the following condition to obtain a fair curve.

$$\min\int_0^1 \sum_{i=1}^{n-1} \ddot{\boldsymbol{r}}_i^{\,2}(t)\,dt \tag{8}$$

Since the curve is linear with respect to $\{x_i\}_{i=1}^{n+2}$,

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the condition leads to a linear equation system. We have got the following one by a symbolic manipulation system.

$$CX=D, \qquad (9)$$

Where each element C_{ij} of C is the product $A_{ij}B_{ij}$ of the corresponding elements of the following A and B. The matrices A, B and C are symmetric.

$$[C_{ij}]=[A_{ij}B_{ij}], \qquad (10)$$

$$A=\left[\begin{array}{ccccc} a_1 & a_1+1 & a_1 & \cdots & a_1 \\ & a_2 & a_2+1 & \ddots & \vdots \\ & & a_3 & \ddots & a_{n-3} \\ & & & \ddots & a_{n-2}+1 \\ & & & & a_{n-1} \\ \\ \\ \end{array}\right.$$

$$\left.\begin{array}{ccc} a_1 & -a_1 & -a_1 \\ \vdots & \vdots & \vdots \\ a_{n-2} & \vdots & \vdots \\ a_{n-1}+1 & -a_{n-1} & -a_{n-1} \\ a_n-4 & -a_n+4 & -a_n+4 \\ & a_n & 0 \\ & & a_n \end{array}\right],$$

$$B=\begin{bmatrix} \boldsymbol{t}_1\cdot\boldsymbol{t}_1 & \boldsymbol{t}_2\cdot\boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n\cdot\boldsymbol{t}_1 & t_{1x} & t_{1y} \\ & \boldsymbol{t}_2\cdot\boldsymbol{t}_2 & \cdots & \boldsymbol{t}_n\cdot\boldsymbol{t}_2 & t_{2x} & t_{2y} \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & \boldsymbol{t}_n\cdot\boldsymbol{t}_n & t_{nx} & t_{ny} \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix},$$

$$D=\begin{bmatrix} (-7\boldsymbol{p}_1+\beta_1\boldsymbol{p}_2)\cdot\boldsymbol{t}_1 \\ (-15\boldsymbol{p}_1+\beta_2\boldsymbol{p}_2+\beta_1\boldsymbol{p}_3)\cdot\boldsymbol{t}_2 \\ (-12\boldsymbol{p}_1+\beta_3\boldsymbol{p}_2+\beta_2\boldsymbol{p}_3+\beta_1\boldsymbol{p}_4)\cdot\boldsymbol{t}_3 \\ \vdots \\ (-12\boldsymbol{p}_1+\beta_{n-1}\boldsymbol{p}_2+\cdots+\beta_2\boldsymbol{p}_{n-1}+\beta_1\boldsymbol{p}_n)\cdot\boldsymbol{t}_{n-1} \\ (-12\boldsymbol{p}_1+\beta_n\boldsymbol{p}_2+\cdots+\beta_3\boldsymbol{p}_{n-1}+5\boldsymbol{p}_n)\cdot\boldsymbol{t}_n \\ 3(p_{1x}+2p_{2x}+\cdots+2p_{(n-1)x}+p_{nx}) \\ 3(p_{1y}+2p_{2y}+\cdots+2p_{(n-1)y}+p_{ny}) \end{bmatrix},$$

$$X=\begin{bmatrix} x_1 \\ \vdots \\ x_{n+2} \end{bmatrix}, \quad \boldsymbol{p}_i=\begin{bmatrix} p_{ix} \\ p_{iy} \end{bmatrix}, \quad \boldsymbol{t}_i=\begin{bmatrix} t_{ix} \\ t_{iy} \end{bmatrix},$$

$$a_1=4, \quad a_i=24(i-1), \quad i=2, \cdots, n,$$
$$\{\beta_1, \beta_2, \cdots, \beta_n\}$$
$$=\{3, -12, -27, -24, \cdots, -24\}.$$

3. Examples

Figure 1 (a) shows a cubic C^2 interpolating curve and a quartic C^2 interpolating curve with the same tangential directions. Both curves are visually and mathematically identical except for numerical errors. Figure 1 (b) shows a quartic C^2 curve with tangential directions rotated by 8, −8, −10 and 10 degrees at 4 points from the left end, respectively. It tells us that we can obtain a desired curve specifying tangential directions just at necessary points and at the others with default values.

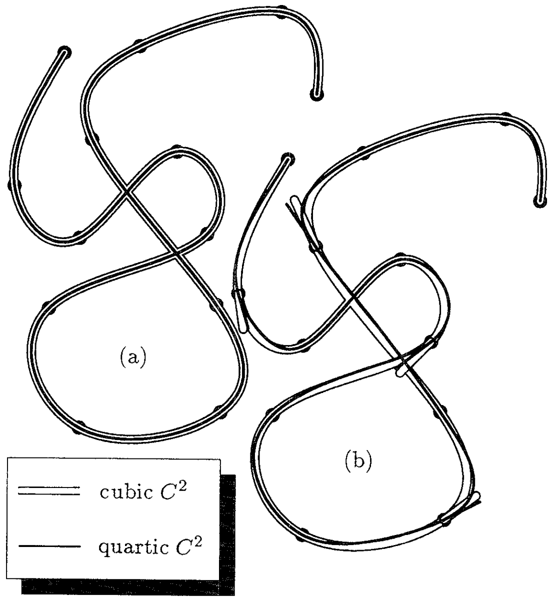

Fig. 1 Cubic C^2 interpolating curve and quartic C^2 interpolating curves (a) with the same tangential directions and (b) with tangential directions rotated at 4 points. These directions are also shown.

4. Concluding Remarks

We have developed a quartic C^2 curve satisfying human-familiar parameters such as passing points and tangential directions, minimizing the integral of square of its second derivative. The other characteristics of the new curve are as follows:

- The curve is easily obtained by solving a linear equation system without variational procedure.
- Specification of the same tangential directions as the cubic C^2 interpolating curve leads to a shape almost as same as the cubic curve.
- Using these tangential directions as defaults, we can obtain our desired curve in respect to specifying tangential directions only at necessary points.

We neglected a similar curve by the $B2$-splines instead of the S-splines because it is slightly complicated and generates a shape almost as same as the curve presented above[4].

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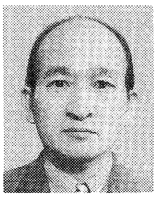

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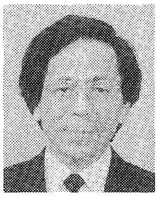

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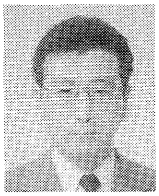

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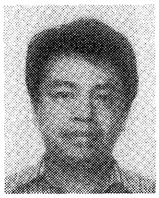

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