Balanced C_{24}-t-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{24} be *the* 24-*cycle*. *The* C_{24}-t-*foil* is a graph of t edge-disjoint C_{24}'s with a common vertex and the common vertex is called *the center of the* C_{24}-t-*foil*. In particular, the C_{24}-2-foil and the C_{24}-3-foil are called *the* C_{24}-*bowtie* and *the* C_{24}-*trefoil*, respectively. When K_n is decomposed into edge-disjoint sum of C_{24}-t-foils, it is called that K_n *has a* C_{24}-t-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of C_{24}-t-foils, it is called that K_n *has a balanced* C_{24}-t-*foil decomposition* and this number is called *the replication number*.

2. Balanced C_{24}-t-foil decomposition of K_n

Theorem. K_n has a balanced C_{24}-t-foil decomposition if and only if $n \equiv 1 \pmod{48t}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{24}-t-foil decomposition. Let b be the number of C_{24}-t-foils and r be the replication number. Then $b = n(n-1)/48t$ and $r = (23t+1)(n-1)/48t$. Among r C_{24}-t-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{24}-t-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $2tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/48t$ and $r_2 = 23(n-1)/48$. Therefore, $n \equiv 1 \pmod{48t}$ is necessary.
(Sufficiency) Put $n = 48st+1, T = st$. Then $n = 48T+1$. Construct a C_{24}-T-foil as follows:
$\{ (48T+1, 1, 4T+2, 16T+2, 28T+3, 12T+2, 38T+3, 14T+2, 32T+3, 18T+2, 8T+2, 2T+1, 4T+3, 2T+2, 8T+4, 18T+3, 32T+5, 14T+3, 38T+5, 12T+3, 28T+5, 16T+3, 4T+4, 2)$,
$(48T+1, 3, 4T+6, 16T+4, 28T+7, 12T+4, 38T+7, 14T+4, 32T+7, 18T+4, 8T+6, 2T+3, 4T+7, 2T+4, 8T+8, 18T+5, 32T+9, 14T+5, 38T+9, 12T+5, 28T+9, 16T+5, 4T+8, 4)$,
$(48T+1, 5, 4T+10, 16T+6, 28T+11, 12T+6, 38T+11, 14T+6, 32T+11, 18T+6, 8T+10, 2T+5, 4T+11, 2T+6, 8T+12, 18T+7, 32T+13, 14T+7, 38T+13, 12T+7, 28T+13, 16T+7, 4T+12, 6)$,
$\ldots$,
$(48T+1, 2T-1, 8T-2, 18T, 32T-1, 14T, 42T-1, 16T, 36T-1, 20T, 12T-2, 4T-1, 8T-1, 4T, 12T, 20T+1, 36T+1, 16T+1, 42T+1, 14T+1, 32T+1, 18T+1, 8T, 2T) \}$.
Decompose this C_{24}-T-foil into s C_{24}-t-foils. Then these starters comprise a balanced C_{24}-t-foil decomposition of K_n.

Corollary 1. K_n has a balanced C_{24}-bowtie decomposition if and only if $n \equiv 1 \pmod{96}$.

Corollary 2. K_n has a balanced C_{24}-trefoil decomposition if and only if $n \equiv 1 \pmod{144}$.

Example 1. *Balanced* C_{24}-2-*foil decomposition of* K_{97}.
$\{(97, 1, 10, 34, 59, 26, 79, 30, 67, 38, 18, 5, 11, 6, 20, 39, 69, 31, 81, 27, 61, 35, 12, 2)$,
$(97, 3, 14, 36, 63, 28, 83, 32, 71, 40, 22, 7, 15, 8, 24, 41, 73, 33, 85, 29, 65, 37, 16, 4)\}$.
This stater comprises a balanced C_{24}-2-foil decomposition of K_{97}.

Example 2. *Balanced* C_{24}-3-*foil decomposition of* K_{145}.
$\{(145, 1, 14, 50, 87, 38, 117, 44, 99, 56, 26, 7, 15, 8, 28, 57, 101, 45, 119, 39, 89, 51, 16, 2)$,
$(145, 3, 18, 52, 91, 40, 121, 46, 103, 58, 30, 9, 19, 10, 32, 59, 105, 47, 123, 41, 93, 53, 20, 4)$,
$(145, 5, 22, 54, 95, 42, 125, 48, 107, 60, 34, 11, 23, 12, 36, 61, 109, 49, 127, 43, 97, 55, 24, 6)\}$.
This stater comprises a balanced C_{24}-3-foil decomposition of K_{145}.

Example 3. *Balanced C_{24}-4-foil decomposition of K_{193}.*
$\{(193, 1, 18, 66, 115, 50, 155, 58, 131, 74, 34, 9, 19, 10, 36, 75, 133, 59, 157, 51, 117, 67, 20, 2),$
$(193, 3, 22, 68, 119, 52, 159, 60, 135, 76, 38, 11, 23, 12, 40, 77, 137, 61, 161, 53, 121, 69, 24, 4),$
$(193, 5, 26, 70, 123, 54, 163, 62, 139, 78, 42, 13, 27, 14, 44, 79, 141, 63, 165, 55, 125, 71, 28, 6),$
$(193, 7, 30, 72, 127, 56, 167, 64, 143, 80, 46, 15, 31, 16, 48, 81, 145, 65, 169, 57, 129, 73, 32, 8)\}.$
This stater comprises a balanced C_{24}-4-foil decomposition of K_{193}.

Example 4. *Balanced C_{24}-5-foil decomposition of K_{241}.*
$\{(241, 1, 22, 82, 143, 62, 193, 72, 163, 92, 42, 11, 23, 12, 44, 93, 165, 73, 195, 63, 145, 83, 24, 2),$
$(241, 3, 26, 84, 147, 64, 197, 74, 167, 94, 46, 13, 27, 14, 48, 95, 169, 75, 199, 65, 149, 85, 28, 4),$
$(241, 5, 30, 86, 151, 66, 201, 76, 171, 96, 50, 15, 31, 16, 52, 97, 173, 77, 203, 67, 153, 87, 32, 6),$
$(241, 7, 34, 88, 155, 68, 205, 78, 175, 98, 54, 17, 35, 18, 56, 99, 177, 79, 207, 69, 157, 89, 36, 8),$
$(241, 9, 38, 90, 159, 70, 209, 80, 179, 100, 58, 19, 39, 20, 60, 101, 181, 81, 211, 71, 161, 91, 40, 10)\}.$
This stater comprises a balanced C_{24}-5-foil decomposition of K_{241}.

Example 5. *Balanced C_{24}-6-foil decomposition of K_{289}.*
$\{(289, 1, 26, 98, 171, 74, 231, 86, 195, 110, 50, 13, 27, 14, 52, 111, 197, 87, 233, 75, 173, 99, 28, 2),$
$(289, 3, 30, 100, 175, 76, 235, 88, 199, 112, 54, 15, 31, 16, 56, 113, 201, 89, 237, 77, 177, 101, 32, 4),$
$(289, 5, 34, 102, 179, 78, 239, 90, 203, 114, 58, 17, 35, 18, 60, 115, 205, 91, 241, 79, 181, 103, 36, 6),$
$(289, 7, 38, 104, 183, 80, 243, 92, 207, 116, 62, 19, 39, 20, 64, 117, 209, 93, 245, 81, 185, 105, 40, 8),$
$(289, 9, 42, 106, 187, 82, 247, 94, 211, 118, 66, 21, 43, 22, 68, 119, 213, 95, 249, 83, 189, 107, 44, 10),$
$(289, 11, 46, 108, 191, 84, 251, 96, 215, 120, 70, 23, 47, 24, 72, 121, 217, 97, 253, 85, 193, 109, 48, 12)\}.$
This stater comprises a balanced C_{24}-6-foil decomposition of K_{289}.

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