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Sizes of a Symmetric Coterie

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A coterie is a set of subsets in which any two of the subsets intersect and in which no subset contains any other subset. A subset is called a quorum. Coteries are used in many distributed algorithms such as distributed mutual exclusion and replicated data management. For distributed algorithms, symmetric coteries are desirable because they realize equal distribution of load, responsibility, and information. In this paper, we show that a lower bound of the quorum size of a symmetric coterie, whose quorum sizes are the same and in which each node appears in the same number of quorums, is $\sqrt{n}$, where n is the number of nodes. When a further condition is added, specifying that the size of the intersection of two distinct quorums is always 1, an upper bound of the coterie size of a symmetric coterie is n. It is also shown that a finite projective plane is an optimal symmetric coterie.

1. Introduction

Many distributed algorithms use logical structures. In distributed mutual exclusion, replicated data management, distributed commit protocols, and name service, efficient algorithms are derived by imposing a logical structure such as a tree,[1] a grid,[2)–8] a symmetric balanced incomplete block design including a finite projective plane,[2),5),9),10] a majority group,[11),12] or combinations of them[13)–15] on the physical network. These algorithms use the logical structure to form a quorum set. A node that collects acknowledgments from a quorum, which is a subset of nodes, can perform a certain operation. Most algorithms use the intersection property of a quorum set.

Garcia-Molina and Barbara formalized the quorum set with the intersection property and defined the *coterie*.[16),17] They discussed domination, the maximum number of quorums in a coterie, the relationship with vote assignment, and other properties of coteries. Researchers proposed many methods of constructing coteries and analyzed them in terms of the quorum size, the coterie size, the availability of a coterie, and other measures.[1)–3),7),8),11)–15),18)–23] However, most of their analysis involves one or more instances of particular coteries such as one based on a grid, and there has been little work on the case in which the quorum size and the coterie size are generalized.

In this paper, we give a lower bound of the quorum size of a symmetric coterie in which the quorum sizes are the same and in which each node appears in the same number of quorums. We also investigate a stricter symmetric case, in which the size of the intersection of any two distinct quorums is always 1. We provide an upper bound of the size of a symmetric coterie in such a case. There are two reasons that we concentrate on symmetric coteries. The first is that symmetry is one of the most important properties of distributed systems. Many distributed algorithms use symmetric coteries, because the load, responsibility, and information for the algorithm should be equally distributed to all nodes. When a tree-based coterie, which is asymmetric, is used, the root node has a greater load than the other nodes and can become a bottleneck. Another example is a coterie, $\{\{1\}\}$, under a set of n nodes numbered from 1 to n. When the coterie is used in mutual exclusion, node 1 becomes a central controller and cannot support load balancing in mutual exclusion. Our second reason for concentrating on symmetric coteries is that the quorum size and the coterie size vary widely when asymmetry is allowed; the quorum size of an asymmetric coterie can be varied from 1 to n, and the coterie size can be 1 and can be larger than $2^n/n$. For example, $\{\{1\}\}$ is a coterie in which the quorum size is 1 and the coterie size is 1. The quorum size becomes n when a coterie is $\{\{1, 2, \cdots, n\}\}$. On the other hand, the coterie size of the coterie $\{\{1, 2\}\}, \{1, 3\}, \cdots, \{1, n\}\}$ is $n-1$.

The rest of the paper is organized as follows. Section 2 gives an overview of coteries and our definitions of symmetry. Section 3 gives a lower bound of the quorum size of a symmetric coterie. We show that the quorum size cannot be

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less than $\sqrt{n}$, where n is the number of nodes. We also show that an upper bound of the coterie size of a symmetric coterie is n when the size of the intersection of any two distinct quorums is 1. We also discuss the relationships among the parameters of a symmetric coterie. We give some examples of symmetric coteries and discuss them in Section 4. We show that a finite projective plane is an optimal symmetric coterie in terms of the quorum size and the coterie size. Our conclusions are stated in Section 5.

2. Definitions

2.1 Coteries

We give definitions and theorems on coteries[16] in Section 2.1. A coterie is a set of distinct subsets of nodes such that any two of the subsets have at least one common node and no subset contains any other subset. We provide a formal definition:

Definition 1 *Let U denote the set of nodes numbered from 1 to n. A set of distinct subsets, S, is a coterie under U iff*

1. *$G \in S$ implies that $G \neq \phi$, and that $G \subseteq U$*
2. *If $G, H \in S$, then $G \cap H \neq \phi$*
3. *There are no $G, H \in S$ such that $G \subset H$.*

We call the second property the intersection property, and the third the *minimality*. A subset $G \in S$ is called a *quorum*. Note that not all nodes in U must appear in a coterie. For example, $\{\{1\}\}$ is a coterie under U.

Theorem 1 *The maximum number of quorums in a coterie under U is bounded by 2^{n-1}.*

Theorem 2 *There are coteries that have an exponential number of quorums on n.*

2.2 Conditions for Symmetry

We define two levels of symmetry:

Definition 2 *A coterie is a level-one (L1) symmetric coterie iff*

1. *All nodes in U appear in the coterie,*
2. *The sizes of quorums in the coterie are the same,*
3. *Each node appears in the same number of quorums.*

The size of a quorum is the number of nodes in a quorum. We call it the *quorum size*. We also call the number of quorums in a coterie the *coterie size*.

Definition 3 *A coterie is a level-two (L2) t-symmetric coterie iff*

1. *It is a level-one symmetric coterie,*
2. *The sizes of the intersection of any two distinct quorums are the same, with value t.*

The first condition of the L1 symmetric coterie seems too restrictive; however, the condition can be relaxed when the coterie is applied in practice. For example, let us consider a case in which a system contains x nodes numbered from 1 to x, and in which y nodes numbered from 1 to y among the x nodes work as replicated databases. An L1 symmetric coterie under the set of the y nodes can be created, and all the x nodes use the created coterie. The resulting coterie under the set of x nodes does not satisfy the first condition, because the whole system consists of x nodes; however, the coterie is useful in this case.

Although the definitions of the L1 and L2 symmetric coteries seem to be similar to that of the balanced incomplete block design (BIBD),[24] they are in fact different.

Definition 4 *A (v, b, r, k, λ)-balanced incomplete block design is an arrangement of v distinct objects into b blocks such that each block contains exactly k distinct objects, each object occurs in exactly r different blocks, and every pair of distinct objects a_i, a_j occurs together in exactly λ blocks.*

In a BIBD, two blocks do not always intersect. For example, in the (9, 12, 4, 3, 1)-BIBD

$$\{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\},\{2,5,8\},\{3,6,9\},\{1,5,9\},\{2,6,7\},\{3,4,8\},\{1,6,8\},\{2,4,9\},\{3,5,7\}\},$$

$\{1, 2, 3\}$ and $\{4, 5, 6\}$ do not have any common objects.

A BIBD is called symmetric if $v=b$, which also means $r=k$. A symmetric BIBD (SBIBD) is sometimes described with its parameters, (v, k, λ)-SBIBD. It is known that a finite projective plane of order d is equivalent to a $(d^2+d+1, d+1, 1)$-SBIBD.[24]

3. Sizes of a Symmetric Coterie

3.1 Lower Bound of the Quorum Size

In this section, we first give a lower bound of the quorum size of an L1 symmetric coterie. We then give an upper bound of the coterie size of an L2 1-symmetric coterie.

Since we are dealing with an L1 symmetric coterie, our assumptions are as follows:

1. There are n nodes numbered from 1 to n.
2. The number of quorums in the coterie (the coterie size) is c.
3. The size of a quorum in the coterie (the quorum size) is q.

4. Each node appears in s quorums in the coterie.

We call c quorums in the coterie $Q_1, Q_2, \cdots, Q_c$.

Theorem 3 *A lower bound of a quorum size of an L1 symmetric coterie is* $\sqrt{n}$.

Proof There are c quorums in the coterie and each quorum contains q nodes; hence, the total number of nodes in all quorums is cq. We can count the number of nodes in all quorums in a different way; there are n nodes and each node appears in s quorums. Therefore,

$$cq = ns. \tag{1}$$

Since any two quorums intersect,

$$|Q_i \cap Q_j| \geq 1. \tag{2}$$

Because the size of a quorum is always q,

$$|Q_i| = q. \tag{3}$$

From Eqs. (2) and (3),

$$\sum_{i=1}^{c}\sum_{j=1}^{c}|Q_i \cap Q_j| \geq c(c-1)+cq. \tag{4}$$

Since each node appears in s quorums, each node appears in s^2 intersections of Q_i and Q_j when all combinations of Q_i and Q_j are considered. Let τ_i denote the number of times in which node i appears in $Q_i \cap Q_j$ $(1 \leq i, j \leq c)$. Then we get

$$\sum_{i=1}^{c}\sum_{j=1}^{c}|Q_i \cap Q_j| = \sum_{i=1}^{n}\tau_i = ns^2. \tag{5}$$

From Eqs. (4) and (5), we obtain

$$ns^2 \geq c^2 + c(q-1). \tag{6}$$

Equality holds when the L1 symmetric coterie is an L2 1-symmetric coterie. When $n=1$, $\{\{1\}\}$ is the only L1 symmetric coterie. Thus, $q=1$ when $n=1$. It is obvious that $q \geq 2$ when $n \geq 2$. Therefore,

$$ns^2 \geq c^2. \tag{7}$$

Equality holds only when $n=1$. Next, we get

$$s \geq \frac{c}{\sqrt{n}}. \tag{8}$$

From Eqs. (1) and (8), we obtain the result

$$q = \frac{ns}{c} \geq \frac{n}{c} \cdot \frac{c}{\sqrt{n}} = \sqrt{n}. \tag{9}$$

Thus, a lower bound of the quorum size of an L1 symmetric coterie is $\sqrt{n}$, and $q=\sqrt{n}$ holds only when $n=1$.

This is an important result. In distributed mutual exclusion and replicated data management, efforts have been made to create a symmetric coterie with a smaller size of quorums. Proposals include a majority group whose quorum size is $\lceil (n+1)/2 \rceil$,[11),12)] a finite projective plane whose quorum size is almost $\sqrt{n}$,[2)] a grid-based coterie whose quorum size is $2\sqrt{n}-1$,[2),3),14)] and another grid-based coterie whose quorum size is almost $\sqrt{2n}$.[8)] Theorem 3 shows that it is impossible to create an L1 symmetric coterie whose quorum size is less than $\sqrt{n}$. A coterie whose quorum size is smaller than $\sqrt{n}$ can be constructed when the conditions for an L1 symmetric coterie are broken. However, in this case, the constructed coterie becomes asymmetric and some nodes bear a greater load than others. For example, in a tree-based coterie, the quorum size can be $\log n$; however, the root node is included in all the quorums, and therefore the root node becomes a bottleneck.

It is interesting that the lower bound does not depend on c. The relationship between c and q in an L2 1-symmetric coterie is discussed in Section 3.2. However, in an L1 symmetric coterie, there is no relationship between c and q except Eq. (1).

3.2 Upper Bound of the Coterie Size

We gave a lower bound of the quorum size of an L1 symmetric coterie in Section 3.1. When we impose on an L1 symmetric coterie the condition that the size of intersection of any two distinct quorums is always 1, we can obtain more results related to symmetric coteries. In Section 3.2, we show that an upper bound of the quorum size of an L2 1-symmetric coterie is n. We also investigate the relationship between the quorum size and the coterie size of an L2 1-symmetric coterie and show that a finite projective plane is an optimal symmetric coterie.

Lemma 1 *In an L2 1-symmetric coterie, every pair of distinct nodes appears in one or no quorum.*

Theorem 4 *An upper bound of the coterie size of an L2 1-symmetric coterie is n.*

Proof From Definition 2, each node appears in s quorums. Let the quorums that contain node i be $Q_{i,1}, Q_{i,2}, \cdots, Q_{i,s}$. Since the quorum size is q,

$$\sum_{j=1}^{s}|Q_{i,j}| = qs. \tag{10}$$

From Lemma 1, nodes other than node i appear in one or no quorum of $Q_{i,1}, Q_{i,2}, \cdots, Q_{i,s}$, because node i appears in every quorum of $Q_{i,1}, Q_{i,2}, \cdots, Q_{i,s}$. Thus, node i appears in s quorums and the other $n-1$ nodes appears in one or no quorum of $Q_{i,1}, Q_{i,2}, \cdots, Q_{i,s}$. Hence, we get

$$\sum_{j=1}^{s}|Q_{i,j}| \leq s+n-1. \tag{11}$$

From Eqs. (10) and (11), we obtain

$$n-1 \geq s(q-1). \tag{12}$$

From Theorem 3, $q > \sqrt{n}$ when $n \geq 2$. Hence, when $n \geq 2$, we get

$$s \leq \frac{n-1}{q-1} < \frac{n-1}{\sqrt{n}-1} = \sqrt{n}+1. \quad (13)$$

from Eq. (12). Equation (13) is also true when $n=1$, because $\{\{1\}\}$ is the only L2 1-symmetric coterie in the case where $n=1$, and its parameters are $n=c=q=s=1$. From Eqs. (1) and (6), and discussion in the proof of Theorem 3, we get

$$ns^2 = cqs = c^2 + c(q-1), \quad (14)$$

because the coterie is an L2 1-symmetric coterie, and therefore,

$$c-1 = q(s-1). \quad (15)$$

From Eqs. (9), (12), (13), and (15), we get

$$c-n \leq s-q < (\sqrt{n}+1) - \sqrt{n} = 1. \quad (16)$$

Since c and n are integers, we obtain

$$c \leq n. \quad (17)$$

Therefore, an upper bound of the coterie size of an L2 1-symmetric coterie is n. □

Corollary 1 *In an L2 1-symmetric coterie, $q \geq s$.*

Proof From Eqs. (1) and (17), we get

$$q \geq s, \quad (18)$$

when a coterie is an L2 1-symmetric coterie. □

In a BIBD, Fisher's inequality

$$b \geq v, \quad (19)$$

$$r \geq k, \quad (20)$$

holds. It is interesting to compare Eqs. (17) and (19), and Eqs. (18) and (20). In an L2 1-symmetric coterie, n is the number of nodes, c is the number of quorums in a coterie, q is the number of nodes in a quorum, and s is the number of quorums in which a node appears. In a BIBD, v is the number of objects, b is the number of blocks in a design, k is the number of objects in a block, and r is the number of blocks in which an object appears. Thus, n of an L2 1-symmetric coterie corresponds to v of a BIBD, c to b, q to k, and s to r. In a BIBD, the equation

$$bk = vr, \quad (21)$$

which corresponds to Eq. (1), holds, as does the equation

$$r(k-1) = \lambda(v-1), \quad (22)$$

which is similar to Eq. (12).[24] However, the directions of the inequality signs in Eqs. (17) and (19) are opposite. The same is true for Eqs. (18) and (20).

Theorem 5 *In an L2 1-symmetric coterie, the larger the quorum size becomes, the smaller the coterie size becomes, when n is fixed.*

Proof As we have shown, there is only one L1 symmetric coterie in which $n=c=q=s=1$, when $n=1$. The coterie is also an L2 1-symmetric coterie, because there is only one quorum in the coterie and no intersection of two quorums exists. Therefore, the theorem is true when $n=1$. Hence, we assume that $n \geq 2$, and thus $q > \sqrt{n}$.

From Eqs. (1) and (6), and since the size of the intersection of any two distinct quorums is always 1, we get

$$n\left(\frac{cq}{n}\right)^2 = c^2 + c(q-1), \quad (23)$$

which gives

$$c = \frac{n(q-1)}{q^2-n}. \quad (24)$$

When we define $f(q)$ by

$$f(q) = \frac{n(q-1)}{q^2-n}, \quad (25)$$

we obtain

$$f'(q) = -\frac{n((q-1)^2+(n-1))}{(q^2-n)^2}. \quad (26)$$

Since $n \geq 2$ and $q > \sqrt{n}$,

$$f'(q) < 0. \quad (27)$$

Thus, the theorem is also true when $n \geq 2$. □

As can be seen from Eq. (24), the coterie size, c, can be calculated from the quorum size, q, and the number of nodes n, not from the number of quorums, s, in which a node appears. When n is given, and c and q are decided, then s is determined by Eq. (1).

In L2 1-symmetric coteries, when n and one of c, q, and s are given, the other two parameters can be calculated from Eqs. (1) and (24). However, since all the parameters must be integers and Eqs. (1) and (24) are necessary conditions, it is difficult to find valid parameters of L2 1-symmetric coteries.

Corollary 2 *A lower bound of the quorum size of an L2 1-symmetric coterie is*

$$\left(1+\sqrt{1+4(n-1)}\right)/2.$$

Proof From Eqs. (17) and (24), we obtain

$$q \geq \frac{1+\sqrt{1+4(n-1)}}{2}. \quad (28)$$

This lower bound is larger than $\sqrt{n}$ when $n \geq 2$. The lower bound is equal to $\sqrt{n}$ when $n=1$.

For distributed mutual exclusion and replicated data management algorithms, it is preferable that a coterie should have a smaller quorum size, because the quorum size determines the number of messages required for the algorithm. A coterie with a large number of quorums is also preferable, because it makes a coterie more available. Hence, in an optimal symmetric coterie,

1. The size of a quorum is as small as possible,
2. The number of quorums is as large as possible.

Definition 5 *An L2 1-symmetric coterie is optimal when its quorum size is minimum and its coterie size is maximum.*

Theorem 6 *A finite projective plane is an optimal L2 1-symmetric coterie.*

Proof From Theorems 4 and 5, an optimal L2 1-symmetric coterie is an L2 1-symmetric coterie in which $c=n$. When a finite projective plane is used as a coterie, it becomes an L2 1-symmetric coterie that satisfies $c=n$. Its quorum size achieves the lower bound in Corollary 2. Therefore, a finite projective plane is an optimal symmetric coterie. □

We conjecture that the quorum size of an L2 symmetric coterie becomes larger when the size of intersection of two distinct quorums becomes larger. In many cases, the quorum size determines the number of messages required for a distributed algorithm. Hence, an L2 1-symmetric coterie is an optimal coterie in terms of the number of messages if our conjecture is true.

4. Examples of Symmetric Coteries

In this section, we look at five examples of symmetric coteries. They are all L1 symmetric coteries, and two of them are L2 1-symmetric coteries.

4.1 Majority Group

A set of all distinct subsets that contain $\lceil (n+1)/2 \rceil$ nodes becomes a coterie. We call it the majority group. This is created by using the voting method.[11),12)] In a majority group,

$$c={}_{n}C_{\lceil \frac{n+1}{2} \rceil} \tag{29}$$

$$q=\left\lceil \frac{n+1}{2} \right\rceil \tag{30}$$

$$s=\frac{1}{n}\left\lceil \frac{n+1}{2} \right\rceil \cdot {}_{n}C_{\lceil \frac{n+1}{2} \rceil}. \tag{31}$$

A majority group is not an L2 t-symmetric coterie, because the size of the intersection of two distinct quorums is between 1 and $\lceil (n+1)/2 \rceil -1$.

4.2 Finite Projective Plane

Maekawa first used a finite projective plane in a quorum-based algorithm.[2)] A finite projective plane exists when

$$n=p^2+p+1 \qquad (p \text{ is a power of a prime}). \tag{32}$$

In a finite projective plane,

$$c=p^2+p+1=n \tag{33}$$

$$q=p+1=\lceil \sqrt{n} \rceil \tag{34}$$

$$s=p+1=\lceil \sqrt{n} \rceil. \tag{35}$$

This is an L2 1-symmetric coterie. In a finite projective plane, c realizes the upper bound in Theorem 4, and q realizes the lower bound in Corollary 2.

4.3 Maekawa's Grid-Based Coterie

Maekawa proposed a coterie construction method using a grid.[2)] Let us arrange n nodes in a $\sqrt{n} \times \sqrt{n}$ grid. A quorum contains all nodes in any one row and all nodes in any one column. In this coterie,

$$c=n \tag{36}$$

$$q=2\sqrt{n}-1 \tag{37}$$

$$s=2\sqrt{n}-1. \tag{38}$$

This is not an L2 t-symmetric coterie, because the size of the intersection of any two distinct quorums is either two or $\sqrt{n}$.

4.4 Cheung, Ammar, and Ahamad's Grid-Based Coterie

Cheung, Ammar, and Ahamad proposed another grid-based coterie.[3)] They also used the $\sqrt{n} \times \sqrt{n}$ grid. A quorum contains all nodes in any one column and one node in each of the other columns. In this coterie,

$$c=\sqrt{n}^{\sqrt{n}} \tag{39}$$

$$q=2\sqrt{n}-1 \tag{40}$$

$$s=\frac{1}{n}\sqrt{n}^{\sqrt{n}}(2\sqrt{n}-1). \tag{41}$$

This is not an L2 t-symmetric coterie either, because the size of the intersection of two distinct quorums is between two and $2\sqrt{n}-2$.

4.5 Agrawal and Jalote's Grid-Based Coterie

Agrawal and Jalote constructed a symmetric coterie.[8)] Their coterie exists when

$$n=\frac{e(e-1)}{2} \qquad (e \text{ is a positive integer}). \tag{42}$$

Let arrange n nodes in an $e \times e$ matrix, G, such that

$$g_{i,j}=\begin{cases} e(i-1)-\frac{1}{2}i(i+1)+j, & \text{for } i<j \\ \phi, & \text{for } i \geq j \end{cases} \tag{43}$$

We then create e quorums such that the i-th quorum contains all nodes in the i-th column and all nodes in the i-th row. In this coterie,

$$c=e=\lceil \sqrt{2n} \rceil \tag{44}$$

$$q=e-1=\lceil \sqrt{2n} \rceil -1 \tag{45}$$

$$s=2. \tag{46}$$

This is an L2 1-symmetric coterie.

4.6 Discussion

Let us look at two L2 1-symmetric coteries, a finite projective plane and Agrawal and Jalote's coterie. The quorum size of the latter is larger than that of the former and the coterie size of the latter is smaller than that of the former. This is what we showed in Theorem 5. The relationship between the quorum size and the coterie size does not hold in the case of L1 symmetric coteries. In both Maekawa's grid-based coterie and Cheung, Ammar, and Ahamad's grid-based coterie, the quorum size is $2\sqrt{n}-1$. However, the former has n quorums in the coterie, and the latter has $\sqrt{n}^{\sqrt{n}}$ quorums in the coterie. The latter is a better coterie because it has higher availability. In both coteries, the number of quorums in which a node appears is calculated from Eq. (1). As we stated in Section 3, in L2 1-symmetric coteries, when n and one other parameter are given, the other two parameters are determined automatically. This is not true in L1 symmetric coteries. For example, whereas Maekawa's grid-based coterie and Cheung, Ammar, and Ahamad's grid-based coterie have the same value for n and q, they have different values for c and s.

As stated in Theorem 4, an upper bound of the coterie size of an L2 1-symmetric coterie is n; however, we can construct an L1 symmetric coterie with a larger coterie size. For example, the majority group has $\lceil (n+1)/2 \rceil$ quorums, and Cheung, Ammar, and Ahamad's coterie has $\sqrt{n}^{\sqrt{n}}$ quorums. In a majority group, in Maekawa's grid-based coterie, and in Cheung, Ammar, and Ahamad's grid-based coterie, the sizes of the intersection of two distinct quorums are not the same. In this case, the cost of an algorithm for distributed mutual exclusion sometimes depends on which nodes issue requests. Hence, there is a trade-off between symmetry and coterie size.

Among the five symmetric coteries in Section 4, some of the majority groups and some of the finite projective planes are ND-coteries.[16] However, the quorum size of the majority group is large. Though a tree-based coterie[1] is an ND-coterie,[19] it is not good with respect to symmetry, because the root node is contained in each quorum and has a heavy load. There are few coteries that satisfy all the desirable conditions: symmetry, nondomination, small quorum size, and large coterie size. An example of such a good coterie is a finite projective plane of order 2, whose parameters are $n=c=7$, $q=s=3$. This is also an ND-coterie.

5. Conclusions

We have defined L1 and L2 symmetric coteries; an L1 symmetric coterie is a coterie whose quorum sizes are the same and in which each node appears in the same number of quorums, and an L2 t-symmetric coterie is an L1 symmetric coterie in which the size of the intersection of any two distinct quorums is t. We have shown that a lower bound of the quorum size of an L1 symmetric coterie is $\sqrt{n}$. L1 symmetric coteries are used in distributed algorithms such as distributed mutual exclusion and replicated data management. The quorum size is an important measure of a symmetric coterie, because it determines the number of messages required for the algorithm. Our theorem implies that it is impossible to create a fully distributed quorum-based algorithm that requires less than $\sqrt{n}$ messages.

We have also shown that an upper bound of the coterie size of an L2 1-symmetric coterie is n. If a coterie is asymmetric, the coterie size can be larger than $2^n/n$. Generally speaking, the larger the coterie size is, the higher the availability of the coterie is. Therefore, it is useful to know an upper bound of the coterie size. We have discussed the relationship among the parameters of an L2 1-symmetric coterie; as the quorum size increases, the coterie size decreases. Thus, the optimal L2 1-symmetric coterie is the one with the smallest quorum size and the largest coterie size. We have shown that a finite projective plane is an optimal L2 1-symmetric coterie. We also examined our results by discussing five examples of symmetric coteries.

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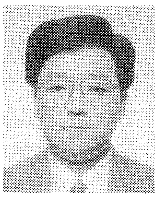

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