A High Accuracy Version of the Yee Algorithm Based on Nonstandard Finite Differences

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1. Introduction: Nonstandard Finite Differences

In one dimension the standard central finite difference (SFD) approximation to a derivative is

$$\frac{d}{dx}f(x) \cong d_x f(x) \equiv \frac{f(x+h/2)-f(x-h/2)}{h} \equiv \frac{\tilde{d}_x f(x)}{h}, \quad (1)$$

where the difference operator, $\tilde{d}_x$, is defined for convenience. Since the solutions of many differential equations can be expanded in terms of sets of basis functions, it is sometimes possible to improve the accuracy of a FD algorithm by defining a NSFD approximation to the derivative in the form

$$\frac{d}{dx}f(x) \cong \frac{\tilde{d}_x f(x)}{s(h)}, \quad (2)$$

where s is "correction function" that minimizes the error $|(\partial_x - d_x)f(x)|$ with respect to a set of basis functions. One might naively try

$$s = \frac{\tilde{d}_x f(x)}{f'(x)}, \quad (3)$$

but this is not always correct. Even when this construct is valid, however, it does not always yield a useful algorithm. In general s must satisfy $\lim_{h\to 0} s(h) = 0$. Since solutions to the wave equation and Maxwell's equations, can be expressed as a Fourier series of the form

$$\psi(\mathbf{x},t) = \sum_{\mathbf{k}} a(\mathbf{k}) e^{i(\mathbf{k}\bullet\mathbf{x}-\omega t)},$$

we can try to improve the accuracy of the FD approximation with respect to these basis functions. Here $k(\mathbf{x}) = \omega/v(\mathbf{x})$, ω is the angular frequency, and v is the phase velocity at $\mathbf{x} = (x,y,z)$. In one dimension (3) is valid, with respect to the basis functions $e^{i(kx\mp\omega t)}$ and we find that

$$s(k,h) = \frac{2}{k}\sin\left(kh/2\right). \quad (4)$$

Using (2) and (4) in the wave equation,

$$\left(\partial_{tt} - v(\mathbf{x},t)^2\nabla^2\right)\psi(\mathbf{x},t) = 0,$$

and replacing the derivatives with NSFDs, an exact FD algorithm is given by

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$$\psi(x,t+\Delta t) = 2\psi(x,t) - \psi(x,t-\Delta t) + u(x)^2 \tilde{d}_x^2 \psi(x,t), \quad \text{where } u(x) = \frac{\sin(\omega\Delta t/2)}{\sin(k(x)h/2)},$$

and $\tilde{d}_x^2 f(x) = f(x+h) + f(x-h) - 2f(x)$. SFD and NSFD solutions of a one-dimensional scattering problem are compared in Ref. 1.

2. High Accuracy Realization of the Yee Algorithm

NSFDs can be defined in more than one dimension and used to solve Maxwell's equations,

$$\mu(x)\partial_t H(x,t) = -\nabla \times E(x,t) \qquad (5a)$$

$$\varepsilon(x)\partial_t E(x,t) = \nabla \times H(x,t). \qquad (5b)$$

Form (2) cannot, however, be simply generalized to higher dimensions. For example, a three-dimensional NSFD approximation to ∂_x is given by

$d_x^{(0)} = \frac{1}{s(k,h)}\tilde{d}_x^{(0)}$, where $\tilde{d}_x^{(0)} = \sum_{i=1}^{3} \alpha_i \tilde{d}_x^{(i)}$. The $\alpha_i = \alpha_i(k)$ are weighting functions that depend on the local wavelength. $\tilde{d}_x^{(1)} = \tilde{d}_x$ is the same as the difference operator in (1), while $\tilde{d}_x^{(2)}$ and $\tilde{d}_x^{(3)}$ are specialized operators that are described in Ref. 2

A high accuracy version of the Yee algorithm can be constructed by simply substituting $d_\xi^{(0)}$ for the spatial partial derivatives in (5a), where $\xi = x, y, z$. For the spatial derivatives in (5b) we replace the partial derivatives by $d_\xi / s(k,h)$, while all time derivatives are replaced by $d_t / s(\omega, \Delta t)$.

3. Results and Conclusions

We have used this algorithm to simulate electromagnetic Mie scattering off dielectrics and conductors. Using cylindrical-shaped objects, we have compared numerical and analytical calculations in two-dimensions. The NSFD gives excellent results even on a coarse grid with a discretization of only $\lambda/h = 8$. The accuracy is limited mainly by the imperfect representation of curved surfaces on the uniform rectangular grid. We have three-dimensional NSFD simulations which we will report later.

References:

(1). J.B. Cole "*Generalized Nonstandard Finite Differences and Physical Applications*," Computers in Physics, vol. 12, no. 1, pp.82-87 (Jan.-Feb., 1998).
(2) "*A High Accuracy Realization of the Yee Algorithm Using Non-Standard* Finite *Differences*," IEEE Transactions on Microwave Theory and Techniques, vol. 45, no. 6, pp. 991-996 (June,1997).