4 C − 3

$\bar{S}_k$ - factorization algorithm of symmetric complete multipartite digraphs

Kazuhiko Ushio

Let $K^*_{n_1,n_2,...,n_m}$ denote *the symmetric complete multipartite digraph* with partite sets $V_1, V_2, ..., V_m$ of $n_1, n_2, ..., n_m$ vertices each, and let $\bar{S}_k$ denote *the evenly partite directed star* from a center-vertex to $k-1$ end-vertices such that the center-vertex is in V_i and every $(k-1)/(m-1)$ end-vertices are in V_j $(j = 1, 2, ..., i-1, i+1, ..., m)$. A spanning subgraph F of $K^*_{n_1,n_2,...,n_m}$ is called an $\bar{S}_k$ - *factor* if each component of F is $\bar{S}_k$. If $K^*_{n_1,n_2,...,n_m}$ is expressed as an arc-disjoint sum of $\bar{S}_k$ - factors, then this sum is called an $\bar{S}_k$ - *factorization* of $K^*_{n_1,n_2,...,n_m}$.

Notation. Given an $\bar{S}_k$ - factorization of $K^*_{n_1,n_2,...,n_m}$, let
r be the number of factors
t be the number of components of each factor
b be the total number of components.
Among r components having vertex x in V_i, let r_{ij} be the number of components whose center-vertex is in V_j.
Among t components of each factor, let t_i be the number of components whose center-vertex is in V_i.

Theorem 1. If $K^*_{n_1,n_2,...,n_m}$ has an $\bar{S}_k$ - factorization, then (i) $k-1 \equiv 0 \pmod{m-1}$ and (ii) $n_1 = n_2 = ... = n_m$ for $k = m$ and $n_1 = n_2 = ... = n_m \equiv 0 \pmod{k(k-1)/(m-1)}$ for $k-1 \geq 2(m-1)$.
Proof. Suppose that $K^*_{n_1,n_2,...,n_m}$ has an $\bar{S}_k$ - factorization. Then $b = 2(n_1n_2 + n_1n_3 + ... + n_{m-1}n_m)/(k-1), t = (n_1 + n_2 + ... + n_m)/k, r = b/t = 2(n_1n_2 + n_1n_3 + ... + n_{m-1}n_m)k/(n_1 + n_2 + ... + n_m)(k-1)$. By the definition of $\bar{S}_k$, $k-1 \equiv 0 \pmod{m-1}$. Put $k = (m-1)a+1$.
For a vertex x in V_i, we have $r_{ii}a = n_j$, $r_{ij} = n_j$ $(j \neq i)$, and $r_{i1} + r_{i2} + ... + r_{im} = r$ $(i = 1, 2, ..., m)$. Therefore, we have $n_1 = n_2 = ... = n_m$. Put $n_1 = n_2 = ... = n_m = n$. Then $r_{ii} = n/a$, $r_{ij} = n (j \neq i)$, $b = mn^2/a, t = mn/((m-1)a+1), andr = n((m-1)a+1)/a$.
Moreover, in a factor, we have $at_1 + at_2 + ... + at_{i-1} + t_i + at_{i+1} + ... + at_m = n$ $(i = 1, 2, ..., m)$ and $t_1 + t_2 + ... + t_m = t$. Therefore, we have $t_i = (at-n)/(a-1) = n/((m-1)a+1)$ for $a \geq 2$.
So we have $n_1 = n_2 = ... = n_m$ for $a = 1$ and $n_1 = n_2 = ... = n_m \equiv 0 \pmod{((m-1)a+1)a)}$ for $a \geq 2$.

Theorem 2. If $K^*_{n,n,...,n}$ has an $\bar{S}_k$ - factorization, then $K^*_{sn,sn,...,sn}$ has an $\bar{S}_k$ - factorization.
Proof. Let $K_{q_1,q_2 \oplus q_3 \oplus ... \oplus q_m}$ denote the multipartite digraph with partite sets $U_1, U_2, U_3, ..., U_m$ of $q_1, q_2, q_3, ..., q_m$ vertices such that q_1 start-vertices in U_1 are adjacent to all q_2 end-vertices in U_2 and q_3 end-vertices in U_3,...,and q_m end-vertices in U_m. Then $\bar{S}_k$ can be denoted by $K_{1,a \oplus a \oplus ... \oplus a}$ for $k = (m-1)a+1$. When $K^*_{n,n,...,n}$ has an $\bar{S}_k$ - factorization, $K^*_{sn,sn,...,sn}$ has a $K_{s,sa \oplus sa \oplus ... \oplus sa}$ - factorization. $K_{s,sa \oplus sa \oplus ... \oplus sa}$ has an $\bar{S}_k$ - factorization. Therefore, $K^*_{sn,sn,...,sn}$ has an $\bar{S}_k$ - factorization.

Theorem 3. When $k = m$, $K^*_{n,n,...,n}$ has an $\bar{S}_k$ - factorization.
Proof. Let $V_i = \{v_{i,1}, v_{i,2}, ..., v_{i,n}\}$ $(i = 1, 2, ..., m)$. Construct mn $\bar{S}_k$ - factors F_{ij} $(i = 1, 2, ..., m; j = 1, 2, ..., n)$ as following:
$F_{ij} = \{(v_{i,1}; v_{1,j}, v_{2,j}, ..., v_{i-1,j}, v_{i+1,j}, ..., v_{m,j}), (v_{i,2}; v_{1,j+1}, v_{2,j+1}, ..., v_{i-1,j+1}, v_{i+1,j+1}, ..., v_{m,j+1}), ..., (v_{i,n}; v_{1,j+n-1}, v_{2,j+n-1}, ..., v_{i-1,j+n-1}, v_{i+1,j+n-1}, ..., v_{m,j+n-1})\}$,
where the additions are taken modulo n with residues $1, 2, ..., n$. Then they comprise an $\bar{S}_k$ - factorization of $K^*_{n,n,...,n}$.

Department of Industrial Engineering, Faculty of Science and Technology, Kinki University, Osaka 577-8502, JAPAN. E-mail:ushio@is.kindai.ac.jp

Theorem 4. When k is odd, $k \geq 5$, and $n \equiv 0 \pmod{k(k-1)/2}$, $K^*_{n,n,n}$ has an $\bar{S}_k$ - factorization.
Proof. Put $k = 2a+1$, $n = s(2a+1)a$, and $N = (2a+1)a$. When $s = 1$, let $V_1 = \{1, 2, ..., N\}$, $V_2 = \{1', 2', ..., N'\}$, and $V_3 = \{1'', 2'', ..., N''\}$. Construct $(2a+1)^2$ $\bar{S}_k$ - factors F_{ij} $(i = 1, 2, ..., 2a+1; j = 1, 2, ..., 2a+1)$ as following:
F_{ij}= { $((A+1);(B+1, ..., B+a)',(C+1, ..., C+a)'')$
$((A+2);(B+a+1, ..., B+2a)',(C+a+1, ..., C+2a)'')$
...
$((A+a);(B+(a-1)a+1, ..., B+a^2)',(C+(a-1)a+1, ..., C+a^2)'')$
$((B+a^2+1)';(A+a+1, ..., A+2a),(C+a^2+1, ..., C+a^2+a)'')$
$((B+a^2+2)';(A+2a+1, ..., A+3a),(C+a^2+a+1, ..., C+a^2+2a)'')$
...
$((B+a^2+a)';(A+a^2+1, ..., A+a^2+a),(C+a^2+(a-1)a+1, ..., C+2a^2)'')$
$((C+2a^2+1)'';(A+a^2+a+1, ..., A+a^2+2a),(B+a^2+a+1, ..., B+a^2+2a)')$
$((C+2a^2+2)'';(A+a^2+2a+1, ..., A+a^2+3a),(B+a^2+2a+1, ..., B+a^2+3a)')$
...
$((C+2a^2+a)'';(A+2a^2+1, ..., A+2a^2+a),(B+2a^2+1, ..., B+2a^2+a)')$ },
where $A = (i-1)a$, $B = (j-1)a$, $C = (i+j-2)a$, and the additions are taken modulo N with residues $1, 2, ..., N$. Then they comprise an $\bar{S}_k$ - factorization of $K^*_{N,N,N}$. Applying Theorem 2, $K^*_{n,n,n}$ has an $\bar{S}_k$ - factorization.

Theorem 5. When $k-1 \equiv 0 \pmod 3$, k is odd, $k \geq 7$, and $n \equiv 0 \pmod{k(k-1)/3}$, $K^*_{n,n,n,n}$ has an $\bar{S}_k$ - factorization.
Proof. Put $k = 3a+1$, $n = s(3a+1)a$, and $N = (3a+1)a$. When $s = 1$, let $V_1 = \{1, 2, ..., N\}$, $V_2 = \{1', 2', ..., N'\}$, $V_3 = \{1'', 2'', ..., N''\}$, and $V_4 = \{1''', 2''', ..., N'''\}$. Construct $(3a+1)^2$ $\bar{S}_k$ - factors F_{ij} $(i = 1, 2, ..., 3a+1; j = 1, 2, ..., 3a+1)$. Then they comprise an $\bar{S}_k$ - factorization of $K^*_{N,N,N,N}$. Applying Theorem 2, $K^*_{n,n,n,n}$ has an $\bar{S}_k$ - factorization.

Theorem 6. When $k-1 \equiv 0 \pmod 4$, $k \equiv 1, 2 \pmod 3$, $k \geq 13$, and $n \equiv 0 \pmod{k(k-1)/4}$, $K^*_{n,n,n,n,n}$ has an $\bar{S}_k$ - factorization.
Proof. Put $k = 4a+1$, $n = s(4a+1)a$, and $N = (4a+1)a$. When $s = 1$, let $V_1 = \{1, 2, ..., N\}$, $V_2 = \{1', 2', ..., N'\}$, $V_3 = \{1'', 2'', ..., N''\}$, $V_4 = \{1''', 2''', ..., N'''\}$, and $V_5 = \{1'''', 2'''', ..., N''''\}$. Construct $(4a+1)^2$ $\bar{S}_k$ - factors F_{ij} $(i = 1, 2, ..., 4a+1; j = 1, 2, ..., 4a+1)$. Then they comprise an $\bar{S}_k$ - factorization of $K^*_{N,N,N,N,N}$. Applying Theorem 2, $K^*_{n,n,n,n,n}$ has an $\bar{S}_k$ - factorization.

References

[1] H. Enomoto, T. Miyamoto and K. Ushio, *C_k - factorization of complete bipartite graphs*, Graphs and Combinatorics, 4 (1988), pp. 111–113.
[2] K. Ushio, *P_3 - factorization of complete bipartite graphs*, Discrete Math. 72 (1988), pp. 361–366.
[3] K. Ushio and R. Tsuruno, *P_3 - factorization of complete multipartite graphs*, Graphs and Combinatorics, 5 (1989), pp. 385–387.
[4] K. Ushio and R. Tsuruno, *Cyclic S_k - factorization of complete bipartite graphs*, Graph Theory, Combinatorics, Algorithms and Applications (SIAM, 1991), pp. 557–563.
[5] K. Ushio, *G - designs and related designs*, Discrete Math. 116 (1993), pp. 299–311.
[6] K. Ushio, *Star-factorization of symmetric complete bipartite digraphs*, Discrete Math. 167/168 (1997), pp. 593–596.
[7] K. Ushio, *$\hat{K}_{p,q}$ - factorization of symmetric complete bipartite digraphs*, To appear in Graph Theory, Combinatorics, Algorithms and Applications (New Issues Press, 1998), pp. 823-826.
[8] K. Ushio, *Cycle-factorization of symmetric complete multipartite digraphs*, To appear in Discrete Math. (1998).
[9] K. Ushio, *$\hat{S}_k$ - factorization of symmetric complete tripartite digraphs*, To appear in Discrete Math. (1998).