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μ-Head Form Proofs with at Most Two Formulas in the Succedent☆

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We investigate a special form of cut-free proofs in classical propositional logic, which we call μ-head form proofs. The number of formula occurrences on the right side of sequents can characterize the distinction between classical systems and intuitionistic systems. From the existence of μ-head form proofs for arbitrary classical propositional theorems, it is derived that at most two occurrences on the right side of sequents are enough to prove them. Moreover, the notion of μ-head form proofs reveals some interesting and intimate connections between classical logic and intuitionistic logic. As a corollary, the μ-head form proofs can be embedded into intuitionistic proofs, a fact well-known as Glivenko's Theorem. The notion of μ-head form proofs separates a proof into a classical part and an intuitionistic part characterized by the disjunction property. Further, this notion can be naturally extended to proofs of a restricted LK, which we call an $L'K$ system. Although the $L'K$ proof is also classical, it contains an intuitionistic proof with a non-trivial form, which satisfies the disjunction property. The $L'K$ system has the cut-elimination property. We can find some analogy between two pairs of sequent calculi $\langle LJ, L'J\rangle$ and $\langle L'K, LK\rangle$.

1. Introduction

In sequent calculi, we can usually distinguish classical systems and intuitionistic systems by a cardinal restriction on the right-hand side of the sequent [18),19)]. This paper reveals that at most two formula occurrences on the right-hand side are enough to prove arbitrary theorems in classical propositional logic. To verify this, we introduce a notion of μ-head form proofs. A simple example of μ-head form proofs of Peirce's law is given below. The following *proof1* will be called a μ-head form proof of $((A \supset B) \supset A) \supset A$ with an invariant A, and *proof2* a μ-head form proof with an invariant $((A \supset B) \supset A) \supset A$. In μ-head form proofs, the right-hand side of each sequent is such that every occurrence on the right side, except for at most one occurrence, is the same as the invariant throughout the proof. From *proof1*, one can easily obtain $\neg A \to ((A \supset B) \supset A) \supset A$ in LJ, and $\to ((A \supset B) \supset A) \supset A, A$ in LK without $(\to c)$. In *proof2*, the right contraction rule is applied only at the end, and the proof is translated into a proof of $\neg(((A \supset B) \supset A) \supset A) \to ((A \supset B) \supset A) \supset A$ in LJ, which is a consequence of Glivenko's theorem.

proof1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A, B}\,(\to w)}{\to A, A \supset B}\,(\to\supset) \qquad A \to A}{(A \supset B) \supset A \to A, A}\,(\supset\to)}{(A \supset B) \supset A \to A}\,(\to c)}{\to ((A \supset B) \supset A) \supset A}\,(\to\supset)$$

proof2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{(A \supset B) \supset A, A \to A}\,(w\to)}{A \to ((A \supset B) \supset A) \supset A}\,(\to\supset)}{A \to ((A \supset B) \supset A) \supset A, B}\,(\to w)}{\to ((A \supset B) \supset A) \supset A, A \supset B}\,(\to\supset) \qquad A \to A}{\dfrac{(A \supset B) \supset A \to ((A \supset B) \supset A) \supset A, A}{\to ((A \supset B) \supset A) \supset A, ((A \supset B) \supset A) \supset A}}}{\to ((A \supset B) \supset A) \supset A}$$

The key notion of invariants shows how to apply the right contraction rules, and it gives a simple embedding into intuitionistic logic.

Definition 1 (μ-Head Form Proofs) A μ-head form proof of $\Gamma \to B$ with an invariant A is defined as a proof of $\Gamma \to B$ such that
(1) the succedent of every sequent in the proof, if not nil, consists only of A except for at most one occurrence; and
(2) in the application of the right logical rules $(\to \neg)$, the parameters in the succedent part, if any appear, consist only of A.

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Let $\Theta[A^{(n)}, B]$ be a sequence consisting of n occurrences of A, where $n \geq 0$, and of at most one occurrence of B, where B is distinct from A. A μ-head form proof of $\Gamma \to \Theta[A^{(n)}, B]$ is similarly defined with the invariant A. A μ-head form proof of $\Gamma \to A$ with an invariant A is called a μ-head form proof of $\Gamma \to A$. ◇

According to the definition, the right-hand side of each sequent of μ-head form proofs consists of at most two kinds of formulae, such that some formulae are the same as the invariant and another (at most one occurrence) can be distinct from it. Every LJ proof can be regarded as a μ-head form proof with empty invariants.

The existence of μ-head form proofs is important not only in formal logic but also in programming based on the notion of proofs-as-programs. The notion of μ-head form proofs makes it possible to construct a binary-conclusion natural deduction system that does not adopt the double-negation elimination rule. The system is a natural extension of intuitionistic natural deduction NJ with at most two consequences [3]. Moreover, μ-head form proofs are useful for embedding classical proofs into intuitionistic proofs [5], and Glivenko's theorem is easily obtained as one of the by-products.

In constructive programming, the Curry-Howard isomorphism [8),11] is known as an important notion for extracting correct programs from constructive proofs of logical specifications. The isomorphism has been extended to proofs of classical logic by Griffin [7], Murthy [10], etc. On the basis of our results in this paper, μ-head form proofs can also be interpreted computationally as exception-handling programs along the line of de Groote [2], and we can successfully use a strict fragment of his system. For each classical theorem A, there always exists an ML-like program of the type A, which contains at most one exception handling [6].

The name of μ-head form proofs comes from Parigot's $\lambda\mu$-calculus [16]. In terms of $\lambda\mu$-calculus, μ-head form proofs can be coded as a form of $\mu\alpha.M$ for some term M such that μ-free variables in M are only α or δ, where α is the name of an invariant, and δ is a special name of $\bot$.

We investigate the statement that if $\Gamma \to B$ in the propositional fragment of LK, then for some A there exists a cut-free μ-head form proof of $\Gamma \to B$ with the invariant A. This paper will show that the statement is valid, and that the consequence B itself can be a witness for the invariant. Since we do not use Glivenko's theorem, which is derived as a corollary, we first consider the problem of calculating truth tables of classical theorems in intuitionistic logic☆. The formula $((A \supset B) \supset A) \supset A$ is a classical theorem. However, $A \to ((A \supset B) \supset A) \supset A$ and $\neg A \to ((A \supset B) \supset A) \supset A$ are intuitionistically derivable. Clearly, then $\Delta \to ((A \supset B) \supset A) \supset A$ is also derivable in intuitionistic logic, where Δ denotes a sequence A, B; $A, \neg B$; $\neg A, B$ or $\neg A, \neg B$. In the appendix, we prove that *if* $\Gamma \to A$ *classically, then* $\Gamma, \Delta \to A$ *intuitionistically for any* Δ *obtained by literals using all the distinct propositional letters in* Γ *and* A☆☆. In the second step, the calculation of truth tables is completed in LK, using cut rules. The cut-elimination process would lead to a cut-free μ-head form proof of A. Therefore, we prove that the cut elimination theorem for μ-head form proofs holds where a restricted LK is used such that at most two formula occurrences appear on the right-hand side of the sequent. As a corollary, μ-head form proofs can be embedded into intuitionistic proofs, a fact well-known as Glivenko's theorem. Moreover, the notion of μ-head form proofs separates a proof into a classical part and an intuitionistic part characterized by the disjunction property.

2. $LK|_2$ System and Cut-Elimination

In this section, we prove the existence of μ-head form proofs of the same conclusion for arbitrary propositional theorems. The following discussion (Theorem 1) is also available for the propositional fragment of LK. However, we define the restricted system $LK|_2$ which has at most two formula occurrences on the right-hand side of each sequent. It is proved that the system is cut-free for μ-head form proofs.

(Axioms)

$$A \to A$$

(Structural Rules)

$$\frac{\Gamma \to A, B}{C, \Gamma \to A, B}\ (w \to) \qquad \frac{\Gamma \to A}{\Gamma \to A, B}\ (\to w)$$

$$\frac{C, C, \Gamma \to A, B}{C, \Gamma \to A, B}\ (c \to) \qquad \frac{\Gamma \to A, A}{\Gamma \to A}\ (\to c)$$

☆ Professor Hiroakira Ono explained this problem.

☆☆ From the referee's report I have learnt that a simple proof of *the statement*: "if $\to A$ in LK, then $\Delta \to A$ in LJ" can be derived by "$\Delta \to A \vee \neg A$ in LJ" *without* the use of Glivenko's theorem.

$$\frac{\Gamma, C, D, \Pi \to A, B}{\Gamma, D, C, \Pi \to A, B}\ (e \to) \quad \frac{\Gamma \to A_2, A_1}{\Gamma \to A_1, A_2}\ (\to e)$$

$$\frac{\Gamma \to A, B \quad B, \Pi \to A, C}{\Gamma, \Pi \to A, C}\ (cut)$$

(Logical Rules)

$$\frac{C, \Gamma \to A, B}{C \wedge D, \Gamma \to A, B}\ (\wedge \to_1)$$

$$\frac{D, \Gamma \to A, B}{C \wedge D, \Gamma \to A, B}\ (\wedge \to_2)$$

$$\frac{\Gamma \to A, B \quad \Gamma \to A, C}{\Gamma \to A, B \wedge C}\ (\to \wedge)$$

$$\frac{C, \Gamma \to A, B \quad D, \Gamma \to A, B}{C \vee D, \Gamma \to A, B}\ (\vee \to)$$

$$\frac{\Gamma \to A, B}{\Gamma \to A, B \vee C}\ (\to \vee_1) \quad \frac{\Gamma \to A, C}{\Gamma \to A, B \vee C}\ (\to \vee_2)$$

$$\frac{\Gamma \to A, B \quad C, \Pi \to A, D}{B \supset C, \Gamma, \Pi \to A, D}\ (\supset \to)$$

$$\frac{B, \Gamma \to A, C}{\Gamma \to A, B \supset C}\ (\to \supset)$$

$$\frac{\Gamma \to A, B}{\neg B, \Gamma \to A}\ (\neg \to) \quad \frac{B, \Gamma \to A}{\Gamma \to A, \neg B}\ (\to \neg)$$

The above rules, except for (axioms), become intuitionistic rules when all the occurrences of A on the right side are nil. Except for (axioms), some of the occurrences on the right-hand side can be nil. In (cut) rules, an occurrence of B that is to be deleted is called a cut-formula. The cut-formula must be literal, i.e., either an atomic formula or a negation of an atom. We define a μ-head form proof of $\Gamma \to A, B$ in $LK|_2$, as in Definition 1.

Theorem 1 ($LK|_2$ Is Cut-Free for μ-Head Form Proofs)
If there are cut-free μ-head form proofs of $\Gamma \to A, B$ and $B, \Pi \to A, C$ with an invariant A, respectively, in $LK|_2$, where B is literal, then there is a cut-free μ-head form proof of $\Gamma, \Pi \to A, C$ with the invariant A in $LK|_2$.

Proof. A rule that infers $\Gamma, \Pi^\flat \to A, C$ from $S_1 : \Gamma \to \Theta$ and $S_2 : \Pi \to A, C$ is called a (mix) rule. Here, the sequences Θ and Π contain common occurrences called m-formulae. The sequence Θ consists of at most two occurrences, namely, the occurrence A (invariant) and the m-formula, where the m-formula must be literal. The sequence $\Pi^\flat$ denotes Π minus all the m-formulae. When S_2 is of the form $\Pi \to C$, where the occurrence C can be nil, (mix) is defined to derive $\Gamma, \Pi^\flat \to \Theta^\flat, C$.

The grade γ of a formula is defined as the number of logical connectives contained in the formula. The left rank ρ_l of a formula B occurring on the right side of a sequent S is defined as the maximal number of consecutive sequents such that S is the lowest of them and that B occurs on the right side in all of them. The right rank ρ_r of a formula is defined similarly. The rank ρ of an m-formula is the sum of ρ_l and ρ_r of the m-formula.

We prove this theorem by the usual double induction on γ and ρ of an m-formula in the lowest (mix) rule, where the m-formula must be literal. Further, we have to check that the invariant must be unchanged even after (mix) elimination. □

Remarks 1 In the $LK|_2$ system, (mix) and (cut) are equivalent.

Corollary 1 (Existence of μ-Head Form Proofs)
If we have $\Gamma \to A$ in the propositional fragment of LK, then there exists a cut-free μ-head form proof of $\Gamma \to A$ in the propositional LK.

Proof. It is remarked that every LJ proof without (cut) can be considered as an $LK|_2$ proof, and that every $LK|_2$ proof can be regarded as an LK proof. According to the appendix, for any Δ_A^Γ we obtain $\Gamma, \Delta_A^\Gamma \to A$ in LJ, which is cut-free by the cut-elimination theorem of LJ and also a μ-head form proof. Assume that there exist n kinds of propositional letters in Γ and A, and let them be $A_1, \cdots, A_n$. Then there are 2^n possibilities of Δ_A^Γ. Among them there exist 2^{n-1} pairs of sequences such that $\neg A_1, \Delta_i$ and A_1, Δ_i for some Δ_i consisting of A_j, where $2 \leq j \leq n$ and $1 \leq i \leq 2^{n-1}$. Apply the cut rule in $LK|_2$ for all the 2^{n-1} pairs of $\Gamma, \Delta_i \to A, \neg A_1$ and $\Gamma, \neg A_1, \Delta_i \to A$ to obtain $\Gamma, \Delta_i \to A$. Hence, $2^n - 1$ applications of the cut rules lead to a μ-head form proof of $\Gamma \to A$, and all the cut rules can be removed by Theorem 1. □

Remarks 2 Since we can also obtain a version of Theorem 1 with respect to the propositional LK instead of $LK|_2$, we can delay applying the right contraction rules until the end in the case of μ-head form proofs in LK, where the right contraction rule is applied only on the invariant of the μ-head form proof.

The following form of statements between two sequent calculi CL and IL, which we call *Key-Lemma1*(CL, IL) and

*Key-Lemma*2 (IL, CL), respectively, plays an important role in our discussion.

Definition 2 (*Key-Lemma*1(*CL*,*IL*)) Let CL and IL be sequent calculi. Let $\Theta[A^{(n)}, B]$ be a sequence consisting of n occurrences of A, where $n \geq 0$, and of at most one occurrence of B, where B is distinct from A. The proposition *Key-Lemma*1 (CL, IL) is defined as the statement that *if there is a* μ*-head form proof of* $\Gamma \to \Theta[A^{(n)}, B]$ *with an invariant* A *in* CL*, then we have* $\Gamma, \neg A \to B$ *in* IL. ◇

In the case of empty invariants, we intend that the conclusion of the statement denotes that *there is a proof of* $\Gamma \to B$ *in* IL.

Definition 3 (*Key-Lemma*2(*IL*,*CL*)) Let IL and CL be sequent calculi. Let $\Gamma/\neg A$ be a sequence Γ minus all $\neg A$'s. The proposition *Key-Lemma*2 (IL, CL) is defined as the statement that *if we have* $\Gamma \to B$ *in* IL*, then* CL *has a* μ*-head form proof of* $\Gamma/\neg A \to A, B$ *with an invariant* A. ◇

If Γ contains no $\neg A$, it is intended that CL *has a* μ*-head form proof of* $\Gamma \to B$ *with empty invariants.* From the definition, a μ-head form proof of $\Gamma \to B$ with empty invariants in LK is identified with an LJ proof. We prove *Key-Lemma*1 (LK, LJ) and *Key-Lemma*2 (LJ, LK) by induction on the derivation.

Lemma 1 *Key-Lemma*1 (LK, LJ) holds.

Proof. Base cases:

1-1. $B \to B$ in LK, where B is not the invariant:
The axiom itself is justified.

1-2. $A \to A$ in LK, where A is the invariant:

$$\frac{A \to A}{A, \neg A \to \ .}\ (\neg \to), (e \to)$$

Step cases:

2-1. Case of $(\to \neg)$, where the principal formula is not the invariant:
We have the μ-head form proof with the invariant A

$$\frac{B, \Gamma \to \Theta[A^{(n)}, nil]}{\Gamma \to \Theta[A^{(n)}, \neg B]\ .}\ (\to \neg)$$

From the induction hypothesis, the LJ proof is obtained as

$$\frac{B, \Gamma, \neg A \to}{\Gamma, \neg A \to \neg B\ .}\ (\to \neg)$$

2-2. Case of $(\to \neg)$, where the principal formula is the invariant:
Let A be $\neg A'$. The assumption gives

$$\frac{A', \Gamma \to \Theta[A^{(n)}, nil]}{\Gamma \to \Theta[A^{(n+1)}, nil]\ .}\ (\to \neg)$$

The induction hypothesis provides the LJ proof

$$\cfrac{\cfrac{\cfrac{A', \Gamma, \neg A \to}{\Gamma, \neg A \to \neg A'}\ (\to \neg)}{\neg A, \Gamma, \neg A \to}\ (\neg \to)}{\Gamma, \neg A \to\ .}\ (e, c \to)$$

The remaining cases of logical rules and structural rules are justified similarly. □

Remarks 3 According to the above proof, we obtain a cut-free LJ proof of $\Gamma, \neg A \to B$ from a cut-free μ-head form proof of $\Gamma \to \Theta[A^{(n)}, B]$ with an invariant A.

Lemma 2 *Key-Lemma*2 (LJ, LK) holds.

Proof. Base cases:

1-1. $B \to B$ in LJ where $B \not\equiv \neg A$:
The axiom itself can be regarded as a μ-head form proof with empty invariant.

1-2. $\neg A \to \neg A$ in LJ:
We can derive the μ-head form proof with the invariant A in LK such that

$$\frac{A \to A}{\to A, \neg A\ .}\ (\to \neg)$$

Step cases:

2-1. Case of $(\to \neg)$, where the side formula is not the form $\neg A$, and Γ contains no $\neg A$:
The assumption

$$\frac{B, \Gamma \to}{\Gamma \to \neg B}\ (\to \neg)$$

itself is the required result.

2-2. Case of $(\to \neg)$, where the side formula is not the form $\neg A$, and Γ contains $\neg A$:
The induction hypothesis gives the μ-head form proof with the invariant A such that

$$\frac{B, \Gamma/\neg A \to A}{\Gamma/\neg A \to A, \neg B\ .}\ (\to \neg)$$

2-3. Case of $(\to \neg)$, where the side formula is the form $\neg A$, and Γ contains no $\neg A$:
The assumption itself is the required proof of $\Gamma \to \neg\neg A$.

2-4. Case of $(\to \neg)$, where the side formula is the form $\neg A$, and Γ contains $\neg A$:
From the induction hypothesis, we have the μ-head form proof with the invariant A

$$\frac{\Gamma/\neg A \to A}{\Gamma/\neg A \to A, \neg\neg A\ .}\ (\to w)$$

3-1. Case of $(\neg \to)$, where the side formula is the form A, and Γ contains no $\neg A$:
The assumption gives the LJ proof

$$\frac{\Gamma \to A}{\neg A, \Gamma \to\ .}\ (\neg \to)$$

From the LJ proof of $\Gamma \to A$, we obtain the μ-head form proof with the invariant A such that

$$\cfrac{\cfrac{\Gamma \to A}{\Gamma \to A, A}\ (\to w)}{\Gamma \to A\ .}\ (\to c)$$

The other cases of $(\neg \to)$ follow a similar pattern. The cases of the remaining logical rules and structural rules can also be confirmed. □

Remarks 4 Following the above proof, we obtain a cut-free μ-head form proof of $\Gamma/\neg A \to A, B$ with an invariant A from a cut-free LJ proof of $\Gamma \to B$.

Corollary 2 (Glivenko's Theorem) For any $n \geq 0$, if we have $\Gamma \to A_1, \ldots, A_n$ in the propositional fragment of LK, then $\Gamma, \neg A_1, \ldots, \neg A_n \to$ in LJ.

Proof. From Corollary 1, there exists a cut-free μ-head form proof of $\Gamma, \neg A_2, \ldots, \neg A_n \to A_1$ in LK. Hence, $\Gamma, \neg A_1, \ldots, \neg A_n \to$ in LJ is derived from *Key-Lemma*1 (LK, LJ). □

Remarks 5 On the other hand, Glivenko's theorem directly derives the existence of μ-head form proofs from *Key-Lemma*2 (LJ, LK).

Corollary 3 (Cut-Elimination) If there is a μ-head form proof of $\Gamma \to \Theta[A^{(n)}, B]$ in LK with an invariant A, where $n \geq 0$, then there is a cut-free μ-head form proof of $\Gamma \to A, B$ with the same invariant A.

Proof. From *Key-Lemma*1 (LK, LJ), the cut-elimination theorem of LJ, and *Key-Lemma*2 (LJ, LK). □

Let $\bigvee \Delta$ be a formula connected by disjunctions for all formulae in a sequence Δ.

Corollary 4 If we have $\Gamma \to \Delta$ in LK, then there is a cut-free and right-contraction-free μ-head form proof of $\Gamma \to \bigvee \Delta, \cdots, \bigvee \Delta$ with an invariant $\bigvee \Delta$ in LK.

Corollary 5 (Disjunction Property) Let Γ consist of Harrop formulae. If there is a μ-head form proof of $\Gamma \to A, B_1 \vee B_2$ with an invariant A in LK, then there is a μ-head form proof of either $\Gamma \to A, B_1$ or $\Gamma \to A, B_2$ with the same invariant A.

Proof. From *Key-Lemma*1 (LK, LJ), the disjunction property of LJ, and *Key-Lemma*2 (LJ, LK). □

Remarks 6
According to *Key-Lemma*1 (LK, LJ), we can decide which subformulae of the given theorem cannot be an invariant of the μ-head form proof. For instance, invariants of μ-head form proofs of Peirce's law, among the subformulae of the theorem, to which $(\to c)$ can be applied, are A and $((A \supset B) \supset A) \supset A$ itself.

3. Notion of μ-Head Form Proofs

Along the lines of Maehara's $L'J$ [9], the notion of μ-head form proofs is naturally extended to obtain the following $L'K$ proofs. We define a sequent calculus system, which we call $L'K$. The inference rules of $L'K$ are the same as those of LK except for the critical inferences. The inferences of $(\to \neg)$ and $(\to\supset)$ are allowed only when the parameters in the succedent part consist only of the same occurrences, which will also be called invariants of the proof. Let Γ and Δ denote finite sequences of formulae, and let $\Theta[A]$ denote a sequence consisting only of A, including nil. In the applications of $(\to\supset)^{**}$ and $(\to \neg)^{**}$, the occurrence of A in $\Theta[A]$ is called an invariant.

(Axioms)

$$B \to B$$

(Structural Rules)

$$\frac{\Gamma \to \Delta}{B, \Gamma \to \Delta}\ (w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, B}\ (\to w)$$

$$\frac{B, B, \Gamma \to \Delta}{B, \Gamma \to \Delta}\ (c \to) \qquad \frac{\Gamma \to \Delta, B, B}{\Gamma \to \Delta, B}\ (\to c)^*$$

$$\frac{\Gamma_1, C, B, \Gamma_2 \to \Delta}{\Gamma_1, B, C, \Gamma_2 \to \Delta}\ (e \to)$$

$$\frac{\Gamma \to \Delta_1, C, B, \Delta_2}{\Gamma \to \Delta_1, B, C, \Delta_2}\ (\to e)$$

$$\frac{\Gamma_1 \to \Delta_1, B \quad B, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}\ (cut)$$

(Logical Rules)

$$\frac{\Gamma \to \Delta, B}{\neg B, \Gamma \to \Delta}\ (\neg \to) \qquad \frac{B, \Gamma \to \Theta[A]}{\Gamma \to \Theta[A], \neg B}\ (\to \neg)^{**}$$

$$\frac{B_i, \Gamma \to \Delta}{B_1 \wedge B_2, \Gamma \to \Delta}\ (\wedge \to_i)$$

$$\frac{\Gamma \to \Delta, B_1 \quad \Gamma \to \Delta, B_2}{\Gamma \to \Delta, B_1 \wedge B_2}\ (\to \wedge)$$

$$\frac{B_1, \Gamma \to \Delta \quad B_2, \Gamma \to \Delta}{B_1 \vee B_2, \Gamma \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to \Delta, B_i}{\Gamma \to \Delta, B_1 \vee B_2}\ (\to \vee_i)$$

$$\frac{\Gamma_1 \to \Delta_1, B \quad C, \Gamma_2 \to \Delta_2}{B \supset C, \Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2} (\supset\to)$$

$$\frac{B, \Gamma \to \Theta[A], C}{\Gamma \to \Theta[A], B \supset C} (\to\supset)^{**}$$

Definition 4 (Maehara's $L'J$) The intuitionistic sequent calculus of Maehara's $L'J$ is redefined as $L'K$ with empty invariants. ◇

With respect to $L'J$, the following theorem is well-known[9),19)].

Theorem 2
(1) $\Gamma \to \Delta$ in $L'J$ if and only if $\Gamma \to \bigvee \Delta^{\star}$ in LJ.
(2) The $L'J$ system has the cut-elimination property.

Definition 5 ($L'K$ Proof with Invariants Φ)
An $L'K$ proof of $\Gamma \to \Delta$ with invariants Φ is defined as a proof of $\Gamma \to \Delta$ obtained by the above system $L'K$, in which every invariant in the proof is in the set Φ. ◇

Definition 6 ($L''K$ Proof with Invariant $\{A\}$)
An $L''K$ proof of $\Gamma \to \Delta$ with an invariant $\{A\}$ is defined as an $L'K$ proof of $\Gamma \to \Delta$ with the invariant $\{A\}$ such that the principal formula of $(\to c)^*$ is the invariant A. ◇

Remarks 7 If $\Gamma \to \Delta$ in $L'K$ with invariants Φ_1 and $\Phi_1 \subseteq \Phi_2$, then $\Gamma \to \Delta$ in $L'K$ with invariants Φ_2. Every $L''K$ proof becomes an $L'K$ proof.

The definitions of *Key-Lemma*1 and *Key-Lemma*2 are naturally extended to the case of a set of invariants. For a set Φ and a sequence Δ, a sequence Δ/Φ is defined as the sequence obtained by deleting all elements in Φ from Δ. A sequence $\neg\tilde{\Phi}$ denotes the sequence consisting of negated formulae in Φ. The following *Key-Lemmata* are proved by induction on the derivation.

Lemma 3 (*Key-Lemma*1($L'K$, $L'J$))
*Key-Lemma*1 $(L'K, L'J)$ holds. That is, if we have $\Gamma \to \Delta$ with invariants Φ in $L'K$, then $\Gamma, \neg\tilde{\Phi} \to \Delta/\Phi$ in $L'J$.

Lemma 4 (*Key-Lemma*1($L''K$, LJ))
*Key-Lemma*1 $(L''K, LJ)$ holds. That is, if we have $\Gamma \to \Delta$ in $L''K$ with an invariant $\{A\}$, then $\Gamma, \neg A \to \bigvee(\Delta/A)$ in LJ.

Lemma 5 (*Key-Lemma*2($L'J$, $L'K$))
*Key-Lemma*2 $(L'J, L'K)$ holds.

Lemma 6 (*Key-Lemma*2(LJ, $L''K$))
*Key-Lemma*2 $(LJ, L''K)$ holds.

Corollary 6 $\Gamma \to \Delta$ in LK if and only if $\Gamma \to \bigvee \Delta$ in $L''K$ with an invariant $\bigvee \Delta$.
With respect to LK and $L'K$, the same corollary holds.

Proof. If-part: From Corollary 2, we obtain $\Gamma, \neg \bigvee\Delta \to$ in LJ. Then by *Key-Lemma*2 $(LJ, L''K)$, $\Gamma \to \bigvee \Delta$ with an invariant $\bigvee \Delta$ is derived in $L''K$. The only-if part is trivial. □

Remarks 8 According to Corollary 6, a classical theorem itself can be an invariant of the proof.

Corollary 7 (Cut-Elimination)
If we have $\Gamma \to \Delta$ in $L'K$ with an invariant $\{A\}$, then there is a cut-free proof of $\Gamma \to \Delta, A$ with the same invariant $\{A\}$ in $L'K$.
With respect to $L''K$, the same corollary on cut-elimination holds.

Proof. From *Key-Lemma*1 $(L'K, L'J)$, the cut-elimination property of $L'J$, and *Key-Lemma*2 $(L'J, L'K)$. □

Corollary 8 If we have $\Gamma \to \Delta$ in LK, then there is a cut-free and right-contraction-free proof of $\Gamma \to \bigvee \Delta, \cdots, \bigvee \Delta$ with an invariant $\bigvee \Delta$ in $L'K$.

Proof. From Corollaries 6 and 7. □

Corollary 9 (Disjunction Property)
Let Γ consist of Harrop formulae. If we have $\Gamma \to B_1, \cdots, B_n$ in $L'K$ with an invariant $\{A\}$, where $n \geq 0$, then for some i, $\Gamma \to B_i, A$ with the invariant $\{A\}$ in $L'K$, where $1 \leq i \leq n$.

Proof. From *Key-Lemma*1 $(L'K, L'J)$, the disjunction property of LJ', and *Key-Lemma*2 $(L'J, L'K)$. □

4. Discussion and Concluding Remarks

We have shown that there exists a special form of cut-free proofs for arbitrary classical propositional theorems, which we call μ-head form proofs. In the μ-head form proof of $\Gamma \to A$, the right-hand side of each sequent consists of the same occurrences as the conclusion A except for at most one occurrence. From the existence of μ-head form proofs, we can derive the well-known Glivenko's theorem. Moreover, in order to prove classical propositional theorems, it is enough to consider a sequent calculus that has at most two occurrences on the right-hand side, such as $LK|_2$. The cut-rules in $LK|_2$ have the restriction such that the cut-formula must be literal. Otherwise, although the cut-rules can be removed, invariants would not be closed under the cut elimination. For instance, con-

☆ $\bigvee \Delta$ is a disjunction of the formulae in Δ.

sider the following $LK|_2$ proof with an invariant $A \supset B$, where $\rho_l \geq 2$, $\rho_r = 1$ and the m-formula is $A \supset B$ which is distinct from C and is not in Γ:

$$\dfrac{\dfrac{A, \Gamma \to A \supset B, B}{\Gamma \to A \supset B, A \supset B} \quad A \supset B, \Pi \to C}{\Gamma, \Pi \to C\ .}\ (\text{mix})$$

The above can be transformed into a mix-free proof of the same sequent, but the invariant becomes C. On the other hand, when the right premiss of the (mix) rule is derived from $B, \Pi_1 \to C$ and $\Pi_2 \to A$, where $\Pi = \Pi_1, \Pi_2$, the above can be transformed into a mix-free proof of $\Gamma, \Pi \to A \supset B, C$ with the same invariant $A \supset B$. The cut-elimination property of Corollary 3 is obtained in this sense (see Corollary 3 in the case n of 0).

A binary-conclusion natural deduction system corresponding to $LK|_2$ is discussed in Fujita[3]. The resulting system is a natural extension of NJ with at most two consequences. In Fujita[4),5)], proofs of classical substructural logics are defined in terms of the restricted $\lambda\mu$-terms. The proofs of four classical substructural logics are embedded into those of intuitionistic logic via μ-head form proofs. Further, a direct translation from arbitrary classical propositional proofs to μ-head form proofs is given according to Glivenko's theorem.

Herbrand's theorem can be regarded as a reduction of predicate logic to propositional logic, which might play an important role in an extension of the notion of μ-head form proofs to predicate logics. Let $B(x_1, \cdots, x_n)$ be a quantifier-free formula, and m be a fixed positive integer. Then it is decidable whether there exist terms t_{ij} where $1 \leq j \leq n$ such that $\bigvee_{i=1}^{m} B(t_{i1}, \cdots, t_{in})$ is tautologous. By Herbrand's theorem, this leads to a contradiction of the fact that predicate logic is undecidable. This observation would mean that the number m of B such that $\exists x_1 \cdots x_n.B(x_1, \cdots, x_n)$ implies that $\bigvee_{i=1}^{m} B(t_{i1}, \cdots, t_{in})$ cannot be bounded recursively[1], and that μ-head form proofs might be impossible in predicate logic. However, the existence of μ-head form proofs can be deduced from Glivenko's theorem. In turn, Kuroda's modified version of Glivenko's theorem of predicate logic makes it possible to extend the notion, with some restrictions, to predicate logic. The modified Glivenko's theorem shows that if we have $\Gamma \to A$ in LK, then $\Gamma^\star, \neg A^\star \to$ in LJ, where $A^\star$ is defined as follows:

(1) $A^\star$ is A for an atomic formula A;
(2) $(\neg A)^\star$ is $\neg A^\star$; (3) $(A_1 \wedge A_2)^\star$ is $A_1^\star \wedge A_2^\star$;
(4) $(A_1 \vee A_2)^\star$ is $A_1^\star \vee A_2^\star$;
(5) $(A_1 \supset A_2)^\star$ is $A_1^\star \supset A_2^\star$;
(6) $(\forall x.A)^\star$ is $\forall x.\neg\neg A^\star$;
(7) $(\exists x.A)^\star$ is $\exists x.\neg\neg A^\star$.

From the LJ proof of $\Gamma^\star, \neg A^\star \to$, we obtain a μ-head form proof of $\Gamma^\star \to A^\star$ in LK, using the following lemma:

Lemma 7 *Key-Lemma*2 (LJ, LK) holds for predicate logic.

Proof. By induction on the derivation. □

The above discussion on the existence of μ-head form proofs has the following abstract structure:

Let L_1, L_2, and L_3 be sequent calculi.
(1) *Key-Lemma*2 (L_1, L_2) holds.
(2) If $\Gamma \to A$ in L_3, then $\psi(\Gamma), \neg\psi(A) \to$ in L_1 via an embedding ψ of L_3 into L_1.

From (1) and (2), there is a μ-head form proof of $\psi(\Gamma) \to \psi(A)$ in L_2 if $\Gamma \to A$ in L_3.

According to the abstract structure, another extension to classical propositional modal logic $S4$[13),14)] can be considered by *Key-Lemma*2 $(IS4, S4)$ and an embedding from $S4$ to $IS4$[15] (the intuitionistic fragment of $S4$).

Since μ-head form proofs are classical proofs and have strong connections to intuitionistic proofs, it would be interesting to investigate the notion of μ-head form proofs in intermediate logics, such as MH[17], since the predicate logic MH satisfies Glivenko's theorem.

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Appendix: Calculating Truth Tables in Intuitionistic Logic

To solve the problem of calculating truth tables in intuitionistic logic, we use the method of tableau proofs in classical propositional logic and intuitionistic propositional logic☆☆☆.

We refer to Nerode and Shore[12] the definitions of and some results on tableau proofs, except for the following two lemmata. The tableau procedure starts with some signed proposition, FA, as the root of the tree, and analyzes it into its components (successors) to check whether any analysis leads to a contradiction, that is, whether both TB and FB appear on any path of the tree for some proposition B, in which case there is no counter-model. It is then concluded that we have refuted the negation of the original assumption A, and so have a tableau proof of A.

We define a classical atomic tableaux (**Fig. 1**) as a binary tree labelled T or F. A finite tableau for classical propositional logic, which is a binary tree labelled with signed (T or F) propositions (entries), is defined as follows:
(1) All atomic tableaux are finite tableaux.
(2) If τ is a finite tableau, P is a path on the τ, E is an entry of τ occurring on P, and τ' is obtained from τ by adding the unique atomic tableau with the root entry E to τ at the end of the path P, then τ' is also a finite tableau.
If $\tau_0, \ldots, \tau_n, \ldots$ is a sequence of finite tableaux such that for each $n \geq 0$, τ_{n+1} is constructed from τ_n by an application of the above (2), then $\tau = \cup\tau_n$ is a tableau.

Let τ be a tableau, P be a path on the τ, and E be an entry occurring on P. E is reduced on P if either the entry E contains no logical connectives; or E is applied by the atomic tableau with the root E, and any immediate successor of E is also reduced on P. P is contradictory if both TA and FA are on P for some proposition A. P is reduced if every entry on P is reduced on P. τ is contradictory if every path through τ is contradictory. A tableau proof for classical propositional logic of a proposition A is a contradictory tableau with the root entry FA.

It is proved in Nerode and Shore[12] that if A is tableau-provable then it is valid (soundness),

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☆☆ The revised version with the same title appeared in *Computer Software*, Vol.14, No.2, pp.71–75 (1997).
☆☆☆ See also the footnote in the introduction.

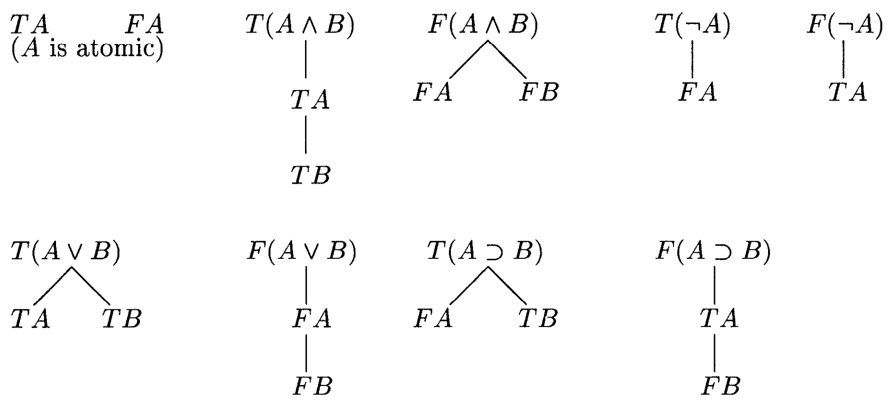

Fig. 1 Classical atomic tableaux.

and that if A is valid then it is tableau provable (completeness).

A tableau proof of A from a set of propositions Γ (premisses) is obtained as an extension of putting TB at the end of every path for every B in Γ.

Lemma 8 If τ is a tableau proof of A from Γ, where every path on the tableau proof is reduced, then for every path P on τ there exists a propositional letter B in A or in some proposition of Γ such that both TB and FB are on P.

Proof. By induction on the structure of A. □

We define an intuitionistic atomic tableau (**Fig. 2**)[12] as a binary tree labelled with signed forcing assertions, that is, $Tp; A$ or $Fq; B$ for propositions A and B, where p and q are in R of a partially ordered set $(\geq, R)$. In the figure, any $q \geq p$ means that we choose any q that appears in an entry and has already been declared greater than or equal to p in $\geq$. Some new $q \geq p$ means that we choose a q not appearing in τ and that it is larger than p in $\geq$. We define an intuitionistic tableau as follows:

(1) Each atomic tableau τ is an intuitionistic tableau.

(2) If τ is a finite intuitionistic tableau, P is a path on τ, E is an entry of τ occurring on P, and τ' is obtained from τ by adjoining an intuitionistic atomic tableau with the root entry E to τ at the end of the path P, then the τ' is also an intuitionistic tableau.

(3) If $\tau_0, \ldots, \tau_n, \ldots$ is a sequence of finite intuitionistic tableau such that τ_{n+1} is constructed from τ_n by an application of the above (2) for every n, then $\tau = \cup\tau_n$ is also an intuitionistic tableau.

Let τ be an intuitionistic tableau, P be a path in τ, and E be an entry on the P. E is reduced on P if either the entry E contains no logical connectives; or E is applied by the atomic tableau with the root E, and any immediate successor of E is also reduced on P. P is contradictory if both $Tp; A$ and $Fp; A$ are on P for some p and A. P is reduced if every entry on P is reduced on P. τ is contradictory if every path through τ is contradictory. Let ϕ be in R. τ is an intuitionistic proof of A if τ is a finite contradictory intuitionistic tableau with its root labelled with $F\phi; A$.

It is proved in Nerode and Shore[12] that if there is an intuitionistic tableau proof of A, then A is intuitionistically valid (soundness), and that if A is intuitionistically valid then it has an intuitionistic tableau proof (completeness).

An intuitionistic tableau proof of A from a set of propositions Γ (premisses) is similarly obtained by putting $T\phi; B$ at the end of every path P for every B in Γ.

For proposition A and premisses Γ, we define Δ_A^Γ as a set consisting of literals obtained by using all the distinct propositional letters in A and Γ.

Lemma 9 If A is classically provable from the set of assumptions Γ, then A is intuitionistically provable from the set of assumptions $\Gamma \cup \Delta_A^\Gamma$ for any Δ_A^Γ.

Proof. Suppose, to generate a contradiction, that A is not intuitionistically provable from $\Gamma \cup \Delta_A^\Gamma$ for some Δ_A^Γ. Then there is an intuitionistic tableau τ' of A from the premisses of Γ, Δ_A^Γ such that some path P' of τ' is not contradictory and that every path on τ' is re-

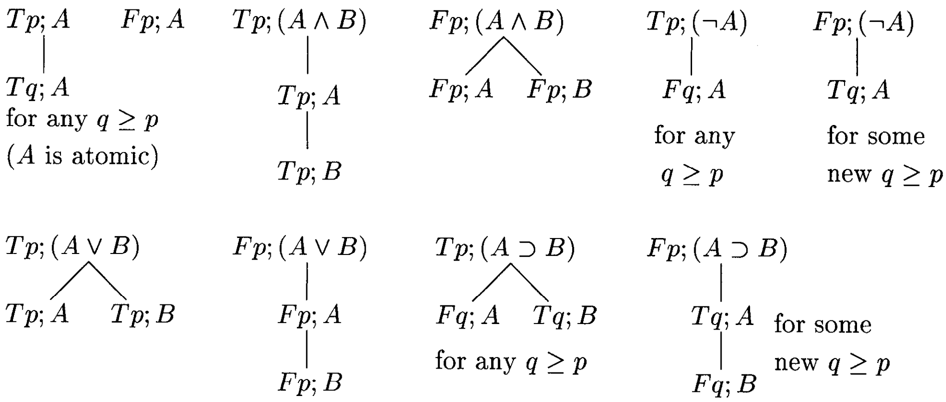

Fig. 2 Intuitionistic atomic tableaux.

duced. Removing all possible worlds (elements in $\geq$) from τ' gives a classical tableau proof τ'' of A from the premisses of Γ, Δ_A^Γ. From Lemma 8, for each path P'' of τ'' there is a propositional letter A_k in Δ_A^Γ such that both TA_k and FA_k are on P''. Hence, also on the path P' of τ', both $Tp_1; A_k$ and $Fp_2; A_k$ exist for some p_1 and p_2 such that P' is not contradictory. However, from the assumption, Δ_A^Γ contains either A_k or $\neg A_k$, and thus both cases of $T\phi; A_k$ and $T\phi; \neg A_k$ lead to a contradiction on P'. □

Now let Γ be a sequence, and let Δ_A^Γ be a sequence consisting of literals obtained by using all the distinct propositional letters in A and Γ. From Lemma 9, if we have $\Gamma \to A$ in LK, then $\Gamma, \Delta_A^\Gamma \to A$ in LJ for any Δ_A^Γ.

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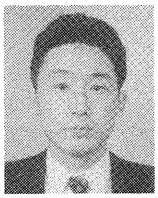

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