Vol. 38 No. 8 Transactions of Information Processing Society of Japan Aug. 1997

Technical Note

The Interpolating Approximation of Smooth Functions on an Interval and by Harmonic Functions

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A new algorithm of the interpolating method is proposed for real and smooth functions defined on a closed interval. The algorithm is a modification of the charge simulation method which has been applied to the functions of a complex valuable.

The charge simulation method is very useful to solve partial differential equations in electrical engineering and has been studied by many researchers[7].

The method has been applied to the numerical conformal mappings by Amano[1]. Inoue[3] has recently proposed a new scheme of approximations different from Amano's. In this paper, the scheme is modified and applied to compute the approximations of real and smooth functions defined on a bounded (or unbounded) interval.

A numerical example is shown to estimate the accuracy of the new method and to help understanding by readers.

Let G and G' denote interior domains whose boundaries are Jordan curves γ and γ', respectively. Without loss of generality, we assume that G and G' contain 0 and ∞ in their interiors and exteriors, respectively.

Let $g(z)$ map conformally G onto G' with

$$g(z) = dz + \cdots, \quad d > 0 \tag{1}$$

near $z = 0$.

Let $g_n(z)$ be the approximation of $g(z)$. Applying asymptotic theorems[2),5),6)] on extremal weighted polynomials, the scheme of the approximation $\log|g_n(z)|$ has recently been proposed[3),4)] as follows.

$$\log|g_n(z)| = \alpha_0 + \log|z| + \sum_{i=1}^{n} \alpha_i \log|1 - \frac{z}{z_{n,i}}|, \tag{2}$$

where $\{\alpha_i\}_{i=0}^n$ (called charges) are constant and $\{z_{n,i}\}_{i=0}^n$ (called charge points) are appropriately chosen outside of G. Note that Eq. (2) has a simple logarithmic singularity at the origin.

For a continuous and smooth function $y = f(x)$ defined on a closed interval

$$I_1 = \{x; 0 \leq x \leq a\}, \tag{3}$$

Equation (2) suggests us a scheme of the approximation $f_n(x)$ of $f(x)$ as follows:

$$f_n(x) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log\left|1 - \frac{x}{z_{n,i}}\right|, \tag{4}$$

where $\{\alpha_i\}_{i=0}^n$ are constant and $\{z_{n,i}\}_{i=1}^n$ are complex numbers not on I_1. $\{\alpha_i\}_{i=0}^n$ and $\{z_{n,i}\}_{i=1}^n$ are determined as follows. Consider $n+1$ points $\{x_{n,i}\}_{i=0}^n$ on I_1 such that

$$0 = x_{n,0} < x_{n,1} < \ldots. < x_{n,n} = a. \tag{5}$$

These points are adopted as the collocation points (**Fig. 1**). This means that the approximations $f_n(x)$ satisfy the condition

$$f_n(x_{n,i}) = f(x_{n,i}) \ (i = 0, 1, \ldots, n). \tag{6}$$

Nextly, we choose n charge points

$$z_{n,i} = x_{n,i} + j\rho \ (j = \sqrt{-1}, i = 1, 2, \ldots, n) \tag{7}$$

on

$$I_2 = \{z; Im\{z\} = \rho\} \ (\rho < 0). \tag{8}$$

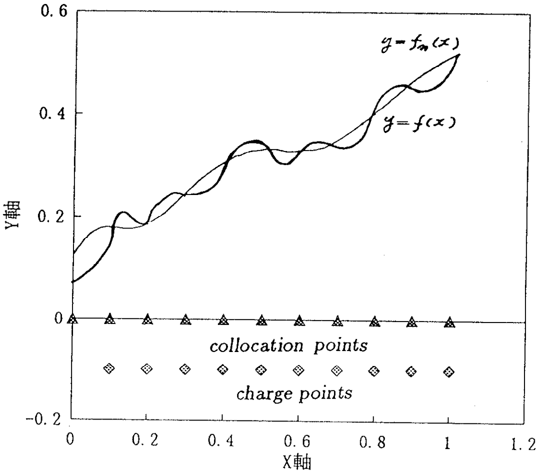

Fig. 1 Collocation and charge points.

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We propose the following algorithm.

Algorithm 1. The approximation $f_n(z)$ of $f(z)$ may be obtained as follows:

(1) The charge points $\{z_{n,i}\}_{i=1}^{n}$ and the collocation points $\{x_{n,i}\}_{i=1}^{n}$ are appropriately chosen on I_2 and on I_1, respectively.

(2) When α_i $(i = 0, 1, 2, \ldots, n)$ are the solutions of a system of simultaneous linear equations

$$\alpha_0 + \sum_{i=1}^{n} \alpha_i \log \left| 1 - \frac{x_{n,k}}{z_{n,i}} \right| = f(x_{n,k}),$$
$$(k = 0, 1, 2, \ldots, n), \qquad (9)$$

the charges at $\{z_{n,i}\}_{i=1}^{n}$ are given by $\{\alpha_i\}_{i=1}^{n}$.

(3) The approximation $f_n(z)$ is represented by

$$f_n(x) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log \left| 1 - \frac{x}{z_{n,i}} \right|. \qquad (10)$$

We note that $f_n(0) = \alpha_0$ follows from Eq. (4). Therefore $\alpha_0 = f(0)$. Since Eq. (2) is a harmonic function in z, the approximation (4) is sufficiently smooth.

The approximation $f_n(x)$ satisfies the "invaiant" property[5], that is: if

$$f_n(x) = f_n(x; z_{n,i}), \qquad (11)$$
$$f_n^*(x) = f_n(ex; ez_{n,i}) \qquad (12)$$

with $e > 0$, then we have $f_n^*(x) = f_n(x)$. Moreover, the approximation of $f(x) + c$ may be obtained by $f_n(x) + c$, where c is constant.

For the approximation $f_n(x)$ of $f(x)$ defined on an unbounded interval

$$I_3 = \{x; x \geq b\} \ (b > 0), \qquad (13)$$

we propose the scheme

$$f_n(x) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log \left| 1 - \frac{z_{n,i}}{x} \right|, \qquad (14)$$

based on Eq. (4) and with a trivial traslation of the charge points $\{z_{n,i}\}_{i=1}^{n}$. Note that Eq. (14) also has the "invariant" property and $\alpha_0 = f(\infty)$.

We call that the schemes (4) and (14) have the "dual" property.

Applying Algorithm 1 for the function

$$y = f(x) = \frac{1}{1 + 25x^2}, \qquad (15)$$

the numerical result is now shown. We note that the function (15) is impossible to be applied the Lagrange interpolating method to compute the approximations of high accuracy, when the collocation points are distributed as follows. Consider the collocation points

$$x_{n,i} = \frac{i}{n} \quad (i = 0, 1, \ldots, n) \qquad (16)$$

on

$$I_1 = \{x; 0 \leq x \leq 1\} \qquad (17)$$

and the charge points

$$z_{n,i} = x_{n,i} + j\rho \quad (j = \sqrt{-1}, i = 1, 2, \ldots, n) \qquad (18)$$

on

$$I_2 = \{z; Im\{z\} = \rho\} \quad (\rho < 0). \qquad (19)$$

The accuracy of the errors is estimated by the maximum of

$$|f_n(x_{n,i+1/2}) - f(x_{n,i+1/2})|, \qquad (20)$$

where

$$x_{n,i+1/2} = \frac{x_{n,i} + x_{n,i+1}}{2}. \qquad (21)$$

The order of the errors are denoted in **Table 1** with parameters n and ρ.

Table 1 Errors for Eq. (15) by Algorithm 1 with n and ρ.

$n \backslash \rho$	−0.1	−0.2	−0.3	−0.4	−0.5
4	4.7E-02	4.5E-02	5.1E-02	5.7E-02	6.1E-02
5	4.6E-02	6.0E-02	6.6E-02	6.9E-02	7.0E-02
10	2.7E-02	2.0E-02	1.4E-02	9.2E-03	5.3E-03
11	2.2E-02	1.4E-02	8.4E-03	4.1E-03	3.3E-03
20	3.0E-03	8.5E-04	6.0E-04	1.2E-03	3.6E-03
21	2.5E-03	7.2E-04	1.7E-03	2.8E-03	2.0E-03

The details of this paper containing the errors estimation at more points will soon appear in a future paper. The numerical calculation has been performed by *Runfort-f77* (PC98-486AV) and single precision.

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(Received January 20, 1997)
(Accepted May 8, 1997)