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Linear Algorithms for a k-partition Problem of Planar Graphs without Specifying Bases

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1 Introduction

In this paper, we consider the following k-partition problem.

Input:

(1) an undirected graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges;

(2) $S \subseteq (V \cup E)(|S| \geq k)$;

(3) k distinct vertices and/or edges $a_i (1 \leq i \leq k) \in S$; and

(4) k natural numbers $n_1, n_2, \ldots, n_k$ such that $\sum_{i=1}^{k} n_i = |S|$.

Output: a partition $S_1 \cup S_2 \cup \ldots \cup S_k$ of the specified set S such that for each $i (1 \leq i \leq k)$

(a) $a_i \in S_i$;

(b) $|S_i| = n_i$; and

(c) there is a connected subgraph $G_i = (V_i, E_i)$ of G such that $S_i \subseteq (V_i \cup E_i)$ and $G_1, G_2, \ldots G_k$ are mutually edge-disjoint.

The problem is called the mixed k-partition problem with respect to edge-disjointness and it is simply called the mixed k-partition problem unless confusion arises. Each a_i is called a base of the subgraph G_i and if all bases are not specified for the mixed k-partition problem, the problem is called the mixed k-partition problem without bases.

In the mixed k-partition problem. if $S = E$ then the problem corresponds to the k-edge-partition problem[3] and if $S \subseteq V$ then the problem corresponds to the k-vertex-partition problem with respect to edge-disjointness [8]. The mixed k-partition problem becomes the k-vertex-partition problem[3, 5] if $S = V$ and the condition "edge-disjointness" in (c) is replaced by "vertex-disjointness".

It has been shown that the k-edge-partition problem and the k-vertex-partition problem with respect to edge-disjointness always have solutions for every k-edge-connected graph [3, 8] and the mixed k-partition problem has a solution for every k-edge-connected graph [9]. Although efficient algorithms are known for these problems provided that k is limited to 2 and 3 [6, 8, 9], no polynomial algorithms are known so far as $k \geq 4$.

On the other hand, if we conider the mixed k-partition problem without bases, the following results have been ontained:

1. For any $k \geq 2$, the mixed k-partition problem without bases can be solved in $O(|V|\sqrt{|V|\log_2|V|} + |E|)$ time for every 4-edge-connected graph $G = (V, E)$.
2. The mixed tripartition problem without bases can be solved in $O(|V|^2)$ time for every 2-edge-connected graph $G = (V, E)$.
3. The mixed 4-partition problem without bases can be solved in $O(|E|^2)$ time for every 3-edge-connected graph $G = (V, E)$.

In this paper, we show that if the input graph is planar, all the mixed k-partition problems stated above can be solved in linear time.

2 The Mixed k-partition Problem Without Bases

Proposition 1 [9] *Let $k \geq 2$. If $G = (V, E)$ has an Eulerian cycle as a spanning subgraph, the mixed k-partition problem without bases can be solved in $O(T_{ec}(G) + |E|)$ time, where $T_{ec}(G)$ is a computation time to find a spanning Eulerian cycle in G.*

A spanning Eulerian cycle G_{ec} for a 4-edge-connected planar graph G can be obtained in linear time as follows.

1. G is transformed to 4-connected graph G' with preserving its planarity.

2. Since G' has a Hamiltonian cycle, we can find a spanning Eulerian cycle of G using the Hamiltonian cycle.

Since this graph transformation can be computed in linear time [7] and a Hamiltonian cycle for a 4-connected planar graph can be computed in linear time [2], we have the following.

Theorem 1 *Let $k \geq 2$. If $G = (V, E)$ is 4-edge-connected planar graph, the mixed k-partition problem without bases can be solved in $O(|E|)$ time.*

3 The Mixed Tripartition and 4-partition Problems Without Bases

Using the graph transformations from k-edge-connected graphs to k-connected graphs, we can show that it is sufficient to show algorithms for k-connected graphs to solve the mixed k-partition problems [9].

3.1 Tripartition

In order to solve the mixed tripartition problem, [9] uses the property of a minimal biconnected graph $G = (V, E)$ such that $G - (x, y)$ has a linear structure of blocks for any edge $(x, y) \in E$. However, it takes $O(|V|^2)$ time to compute the linear structures until the partition is completed. If the input graph is planar, similar linear structures can be found in linear time [7].

Theorem 2 *The mixed tripartition problem for 2-edge-connected planar graph $G = (V, E)$ can be solved in $O(|E|)$ time.*

3.2 4-partition

If $G' = (V', E')$ is a biconnected planar graph and $P = (\{x_i | 0 \leq i \leq \ell\}, \{(x_i, x_{i+1}) | 0 \leq i \leq \ell - 1\})$ is a path graph such that either x_0 or x_ℓ is contained in V', the mixed 4-partition problem without bases can be solved by using Theorem 2 anf the idea shown in [9]. Since any triconnected graph has such structrre [1] and if it is planar such structure can be found in linear time [4], the following theorem is obtained.

Theorem 3 *The mixed 4-partition problem without bases for a 3-edge-connected planar graph $G = (V, E)$ can be solved in $O(|E|)$.*

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References

[1] J. Cherian and S.N. Maheshwari : "Finding nonseparating induced cycles and independent spanning trees in 3-connected graphs," *Journal of Algorithms*, 9, 507-537 (1988).

[2] N. Chiba and T. Nishizeki : "The Hamiltonian Cycle Problem is Linear-Time Solvable for 4-Connected Planar Graphs," *Journal of Algorithms*, 10, 187-211 (1989).

[3] E. Györi: "On division of connected subgraphs," in: Combinatorics(Proc. 5th Hungarian Combinational Coll., 1976, Keszthely) North-Holland, Amsterdam, 485-494 (1978).

[4] L.Jou, H.Suzuki and T.Nishizeki: "A linear algorithm for finding a nonseparating ear decomposition of triconnected planar graphs," Technical paper of IPSJ, AL40-3(1994).

[5] L. Lovász: "A homology theory for spanning trees of a graph," *Acta math. Acad. Sci. Hunger*, 30, 241-251 (1977).

[6] H.Suzuki, N.Takahashi and T.Nishizeki: "A linear algorithm for bipartition of biconnected graphs," *Information Processing Letters*, 33, 5 (1990).

[7] K.Wada and W.Chen: "Linear algorithms for a k-partition problem of planar graphs wihthout specifying bases," Tech. Rep. of Kawaguchi-Lab. of ECE, NIT, TR96-02(1996).

[8] K.Wada and K.Kawaguchi: "Efficient algorithms for triconnected graphs and 3-edge-connected graphs. Proc. of the 19th International Workshop on Graph-Theoretic Concepts in Computer Science(WG'93), Lecture Notes in Computer Science, 790, 132-143(1994).

[9] K.Wada, A.Takaki and K.Kawaguchi: "Efficient algorithms for a Mixed k- partition Problem of Graphs wihthout Specifying Bases," 20th International Workshop on Graph-Theoretic Concepts in Computer Science.(1994), also in Lecture Notes in Computer Science 903, 319-330(1995).