Technical Note

An Interactive Method for Designing Smooth Convex Curves by Using a Cubic B-spline Formulation (II)

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This work is an improved version of the previous method [5] for interactively designing convex curves by using a cubic B-spline formulation. The new algorithm ensures that the highest point of the resulting curve occurs only at that of the ideal curve in the user's mind. Also, it suggests a class of convex B-spline curves that share the same highest point and have different curvatures at the point, providing the user with an easy interactive control of shape.

1. Introduction

Many applications of computer graphics and computer-aided geometric design require to interactively generate convex curves with some level of continuity at end-points. The work of Ref. 5) has introduced an easy-to-use and efficient design method for creating such a curve using a cubic B-spline and a recursive subdivision manner, in the case of given unit tangents at two end-points. The previous method proposed has several properties which may be desirable, but suffers from two problems: (1) it makes the resulting curve have always zero curvature at the sample point selected by the user, and (2) it fails to obtain a satisfactory approximation when the sample point is a considerable distance from the highest point on the ideal curve in the user's mind, where the highest point on a curve is defined to be the point farthest from the line passing through end-points [5].

This paper attempts to improve the previous work in order to cope with these problems. In relevant literature, Goodman and Unsworth [1] presented a shape preserving interpolation scheme to remove the zero-curvature at each of the interpolation points. Piegl [2] discussed how to modify the shape of the nonuniform rational B-spline curves by interactively adjusting control vertices or weights. More recently, Pigounakis and Kaklis [3] developed a two-stage automatic algorithm for fairing C^2-continuous cubic parametric B-splines under convexity, tolerance and end constraints.

The work by Goodman and Unsworth [1] is the closest to our research, but it cannot preserve location of the highest point. The algorithm described here not only lets the user choose as the sample point the highest point on an intended curve, but also leaves it unchanged during interactive manipulation. In addition, it gives the user direct control over shape by moving one point upwards or downwards along a line, through which a class of B-spline curves with different curvatures at the sample point can be created while in the same time maintaining several interesting and useful geometric properties of the original curve.

The remainder of this paper is organized as follows. We begin in Section 2 by reviewing the mathematical formulation of the approximation curve used in this paper. Section 3 shows an improved scheme for the interactive design of convex B-spline curves, and Section 4 gives the example of the application of the scheme. In Section 5, we conclude this paper.

2. Mathematical Formulation

As in the previous work [5], the parametric cubic B-spline chosen to approximate an intended curve is given by

$$\boldsymbol{Q}(t) = \sum_{i=0}^{4} \boldsymbol{B}_i N_{i,4}(t) \qquad 0 \le t \le 2 \qquad (1)$$

where $\boldsymbol{B}_i$, $i = 0, \cdots, 4$ are control vertices, and $N_{i,4}(t)$, $i = 0, \cdots, 4$ are basis functions defined over the knot vector $U = \{0,0,0,0,1,2,2,2,2\}$ by the Cox-deBoor recursion formula [4]

$$N_{i,1}(t) = \begin{cases} 1 & u_i \le t < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - u_i}{u_{i+k-1} - u_i} N_{i,k-1}(t) + \frac{u_{i+k} - t}{u_{i+k} - u_{i+1}} N_{i+1,k-1}(t)$$
$$k = 2,3,4$$

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where the values of u_i are elements of the knot vector U and the convention $0/0 = 0$ is adopted.

The curve $\boldsymbol{Q}(t)$ exhibits (1) the convex hull property: it lies entirely within the convex hull of its control polygon, and (2) convexity-preserving property: it is convex when the control polygon is convex.

3. Curve Design

Suppose we have two end-points $\boldsymbol{P}_0$ and $\boldsymbol{P}_4$ together with the respective unit tangents which intersect at $\boldsymbol{T}$, and the highest point $\boldsymbol{P}_f$ on the ideal curve in the user's mind, as shown in **Fig. 1**. Our goal is to find a convex B-spline curve which lies within the triangle $\boldsymbol{P}_0\boldsymbol{T}\boldsymbol{P}_4$, possesses $\boldsymbol{P}_f$ as its highest point, interpolates $\boldsymbol{P}_0$ and $\boldsymbol{P}_4$, and is tangential at $\boldsymbol{P}_0$ and $\boldsymbol{P}_4$ to the lines $\overline{\boldsymbol{P}_0\boldsymbol{T}}$ and $\overline{\boldsymbol{T}\boldsymbol{P}_4}$, respectively.

The previous design scheme[5] to do this is (1) to select the intersection $\boldsymbol{P}_m$ of the intended curve and the line $\overline{\boldsymbol{T}\boldsymbol{M}}$ as a sample point, where $\boldsymbol{M}$ is the midpoint of $\overline{\boldsymbol{P}_0\boldsymbol{P}_4}$, and (2) to use it as the control vertex $\boldsymbol{B}_2$. As a result, it produces a convex B-spline curve that satisfies the given end conditions and interpolates the sample point $\boldsymbol{P}_m$, where the curvature at $\boldsymbol{P}_m$ is always zero and $\boldsymbol{P}_m$ is the highest point on the curve. Thus, the curve is very flat in the neighborhood of $\boldsymbol{P}_m$. In addition, when $\boldsymbol{P}_m$ is a considerable distance from the highest point $\boldsymbol{P}_f$, the approximation curve and the intended curve have the significant differences.

To better mimic the shape of the intended curve, instead of using $\boldsymbol{P}_m$ as the sample point an alternative that is reasonable and more intuitive is to choose the highest point $\boldsymbol{P}_f$. Thus,

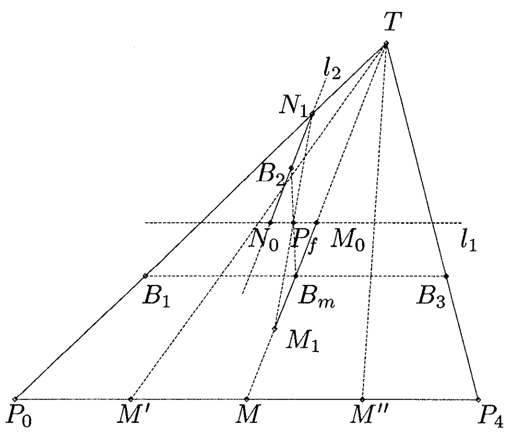

Fig. 1 Construction of a control polygon for a convex B-spline.

by setting 0, 1 and 2 in Eq. (1) to interpolate $\boldsymbol{P}_0$, $\boldsymbol{P}_f$ and $\boldsymbol{P}_4$, and by using the Cox-deBoor formula to calculate basis functions $N_{i,4}(t)$, $i = 0, \cdots, 4$, we obtain

$$\boldsymbol{P}_0 = \boldsymbol{Q}(0) = \boldsymbol{B}_0 \tag{2}$$

$$\boldsymbol{P}_f = \boldsymbol{Q}(1) = \frac{1}{4}\boldsymbol{B}_1 + \frac{1}{2}\boldsymbol{B}_2 + \frac{1}{4}\boldsymbol{B}_3 \tag{3}$$

$$\boldsymbol{P}_4 = \boldsymbol{Q}(2) = \boldsymbol{B}_4 \tag{4}$$

Rearranging the terms in Eq. (3) yields

$$\boldsymbol{B}_2 + \frac{1}{2}(\boldsymbol{B}_1 + \boldsymbol{B}_3) = 2\boldsymbol{P}_f \tag{5}$$

Now define $\boldsymbol{B}_m = \frac{1}{2}(\boldsymbol{B}_1 + \boldsymbol{B}_3)$; i.e., let $\boldsymbol{B}_m$ denote the midpoint of the line connected by $\boldsymbol{B}_1$ and $\boldsymbol{B}_3$. Equation (5) then reduces to

$$\boldsymbol{B}_2 + \boldsymbol{B}_m = 2\boldsymbol{P}_f \tag{6}$$

Geometrically, the equation states that $\boldsymbol{P}_f$ must be the midpoint of the line connected by $\boldsymbol{B}_m$ and $\boldsymbol{B}_2$.

On the other hand, according to the first derivatives of the basis functions at $t = 0$, 1, and 2, we can find that the respective tangents at $\boldsymbol{P}_0$, $\boldsymbol{P}_f$ and $\boldsymbol{P}_4$ are

$$d\boldsymbol{Q}/dt|_{t=0} = 3(\boldsymbol{B}_1 - \boldsymbol{P}_0) \tag{7}$$

$$d\boldsymbol{Q}/dt|_{t=1} = \frac{3}{4}(\boldsymbol{B}_3 - \boldsymbol{B}_1) \tag{8}$$

$$d\boldsymbol{Q}/dt|_{t=2} = 3(\boldsymbol{P}_4 - \boldsymbol{B}_3) \tag{9}$$

It is easy to see from Eqs. (7) and (9) that $\boldsymbol{B}_1$ and $\boldsymbol{B}_3$ lie on the lines $\overline{\boldsymbol{P}_0\boldsymbol{T}}$ and $\overline{\boldsymbol{T}\boldsymbol{P}_4}$, respectively. Also, it is clear from Eq. (8) that the line $\overline{\boldsymbol{B}_1\boldsymbol{B}_3}$ is parallel to $\overline{\boldsymbol{P}_0\boldsymbol{P}_4}$ since $\boldsymbol{P}_f$ is required to be the highest point on the approximation curve. Thus, $\boldsymbol{B}_m$ should be on the midline $\overline{\boldsymbol{T}\boldsymbol{M}}$, as shown in Fig. 1. This means that choosing one point on the midline $\overline{\boldsymbol{T}\boldsymbol{M}}$ can determine a control polygon $\boldsymbol{B}_0\boldsymbol{B}_1\boldsymbol{B}_2\boldsymbol{B}_3\boldsymbol{B}_4$, and then produce an approximation curve that satisfies the given end conditions and makes the tangent at $\boldsymbol{P}_f$ parallel to $\overline{\boldsymbol{P}_0\boldsymbol{P}_4}$.

However, such a curve may not be convex. To assure the convexity of the curve, $\boldsymbol{B}_2$ has to lie inside the triangle constructed by $\boldsymbol{B}_1$, $\boldsymbol{B}_3$ and $\boldsymbol{T}$. Equation (6) indicates that $\boldsymbol{B}_2$ is determined by $\boldsymbol{B}_m$ and $\boldsymbol{P}_f$. We now use Fig. 1 to give conditions on $\boldsymbol{B}_m$ and $\boldsymbol{P}_f$ for an approximation curve to be convex. In this figure, the line l_1 passes through $\boldsymbol{P}_f$ and is parallel to $\overline{\boldsymbol{P}_0\boldsymbol{P}_4}$. $\boldsymbol{M}_0$ is the intersection of l_1 and $\overline{\boldsymbol{T}\boldsymbol{M}}$. $\boldsymbol{N}_0$ is on the line l_1 and $\boldsymbol{N}_0 = 2\boldsymbol{P}_f - \boldsymbol{M}_0$. The line l_2 passes through $\boldsymbol{N}_0$ and is parallel to $\overline{\boldsymbol{T}\boldsymbol{M}}$. $\boldsymbol{N}_1$ is the intersection of l_2 and $\overline{\boldsymbol{P}_0\boldsymbol{T}}$. $\boldsymbol{M}_1$ is the intersection of $\overline{\boldsymbol{TM}}$ and the line connecting $\boldsymbol{N}_1$ and $\boldsymbol{P}_f$. $\boldsymbol{M}'$ and $\boldsymbol{M}''$ are the midpoints of the line segments $\overline{\boldsymbol{P}_0\boldsymbol{M}}$ and $\overline{\boldsymbol{MP}_4}$, respectively. According to the above definitions with respect to $\boldsymbol{P}_f$, $\boldsymbol{B}_m$, $\boldsymbol{M}_0$, $\boldsymbol{M}_1$, $\boldsymbol{M}'$, $\boldsymbol{M}''$ and $\boldsymbol{T}$, we have the following theorem.

Theorem 1 Given a point $\boldsymbol{P}_f$ inside the triangle $\boldsymbol{M}'\boldsymbol{T}\boldsymbol{M}''$ and a point $\boldsymbol{B}_m$ on the line segment $\overline{\boldsymbol{M}_0\boldsymbol{M}_1}$, the resulting curve is convex.

Proof: Consider a point $\boldsymbol{P}_f$ inside the triangle $\boldsymbol{M}'\boldsymbol{T}\boldsymbol{M}''$. Since $\boldsymbol{N}_0 - \boldsymbol{P}_f = \boldsymbol{P}_f - \boldsymbol{M}_0$, $\boldsymbol{N}_0$ is inside the triangle $\boldsymbol{P}_0\boldsymbol{T}\boldsymbol{P}_4$. On the other hand, for a point $\boldsymbol{B}_m$ on the line segment $\overline{\boldsymbol{M}_0\boldsymbol{M}_1}$, $\boldsymbol{B}_2$ satisfying Eq. (6) lies on the line segment $\overline{\boldsymbol{N}_0\boldsymbol{N}_1}$.

Recall that $\boldsymbol{B}_m$ is the midpoint of the line segment $\overline{\boldsymbol{B}_1\boldsymbol{B}_3}$ parallel to $\overline{\boldsymbol{P}_0\boldsymbol{P}_4}$. Thus, as can be seen from Fig. 1, $\boldsymbol{B}_2$ must lie inside the triangle $\boldsymbol{B}_1\boldsymbol{B}_3\boldsymbol{T}$. That is to say, the control polygon $\boldsymbol{P}_0\boldsymbol{B}_1\boldsymbol{B}_2\boldsymbol{B}_3\boldsymbol{P}_4$ is convex. This completes this proof. □

Of course, if $\boldsymbol{P}_f$ happens to be outside the triangle $\boldsymbol{M}'\boldsymbol{T}\boldsymbol{M}''$, the above method is not applicable. In this case, we employ the subdivision strategy described in Ref. 5) to split the intended curve into two parts at $\boldsymbol{P}_f$ and apply the method for each one. This process can be repeated until a satisfactory curve is obtained.

4. Example

To test the effectiveness, the curve design scheme described above has been applied for the example of Fig. 1. For the sake of a better understanding, we chose as $\boldsymbol{B}_m$ the points $\boldsymbol{M}_0$, $(\boldsymbol{M}_0 + \boldsymbol{M}_1)/2$ and $\boldsymbol{M}_1$, respectively. The resulting curves, together with their control polygons, are shown in **Fig. 2**.

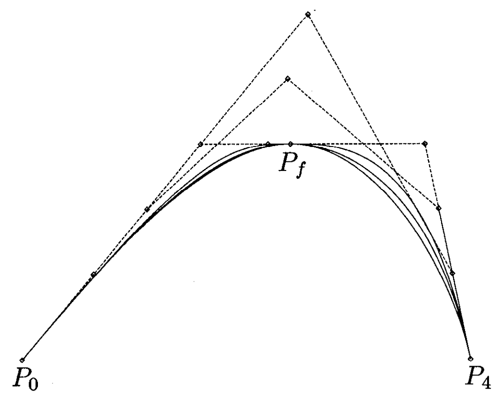

Fig. 2 The convex cubic B-spline curves together with their control polygons.

As can be seen from this figure, all of the three curves have the following properties: convexity preserving, interpolation of the given end conditions, and possession of the highest position at $\boldsymbol{P}_f$. But they have subtle differences in shape. For example, when $\boldsymbol{B}_m = \boldsymbol{M}_0$, the generated curve is flatter in the neighborhood of $\boldsymbol{P}_f$. In fact, the curvatures of them at $\boldsymbol{P}_f$ are 0.0, 0.00473, and 0.00625, respectively. This shows that the curvature at $\boldsymbol{P}_f$ increases as $\boldsymbol{B}_m$ approaches $\boldsymbol{M}_1$. Thus, by moving $\boldsymbol{B}_m$ on $\overline{\boldsymbol{M}_0\boldsymbol{M}_1}$ the user can obtain a family of approximation curves similar to the designed one, among which the most satisfying can be chosen.

5. Concluding Remarks

This paper has presented an improved method for designing a convex B-spline curve in an interactive graphics environment. The algorithm leaves the highest point unchanged in order to reflect the shape characterization of an intended convex curve, and in general allows non-zero curvature at the point. The preliminary experiment indicated that it does indeed provide a convenient geometric means to quickly generate a class of convex B-spline curves with several common geometric properties.

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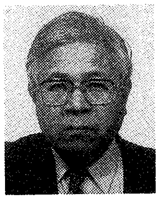

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