S_3-FACTORIZATION ALGORITHM OF COMPLETE BIPARTITE SYMMETRIC DIGRAPHS

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Kazuhiko USHIO
Department of Industrial Engineering
Kinki University, 577 Osaka, Japan

1. Introduction

Let S_3 be a **directed star** with 3 vertices, say u(centervertex), v(endvertex), w(endvertex), and with two directed edges u → v and u → w. Let $K^*_{m,n}$ be a **complete bipartite symmetric digraph** with partite sets V_1 and V_2 of m and n vertices each. A spanning subgraph F of $K^*_{m,n}$ is called an **S_3-factor** if each component of F is isomorphic to S_3. If $K^*_{m,n}$ is expressed as an edge-disjoint sum of S_3-factors, then this sum is called an **S_3-factorization** of $K^*_{m,n}$.

2. S_3-Factor of $K^*_{m,n}$

Theorem 1. $K^*_{m,n}$ has an S_3-factor if and only if (i) $m+n \equiv 0 \pmod 3$, (ii) $2n-m \equiv 0 \pmod 3$, (iii) $2m-n \equiv 0 \pmod 3$, (iv) $m \leqq 2n$ and (v) $n \leqq 2m$.

Proof. Suppose that $K^*_{m,n}$ has an S_3-factor F. Let t be the number of components of F. Then $t=(m+n)/3$. Hence, Condition (i) is necessary. Among these t components, let t_1 and t_2 be the number of components whose endvertices are in V_2 and V_1, respectively. Then, since F is a spanning subgraph of $K^*_{m,n}$, we have $t_1+2t_2=m$ and $2t_1+t_2=n$. Hence $t_1=(2n-m)/3$ and $t_2=(2m-n)/3$. From $0 \leqq t_1 \leqq m$ and $0 \leqq t_2 \leqq n$, we must have $m \leqq 2n$ and $n \leqq 2m$. Conditions (ii)-(v) are, therefore, necessary.
For those parameters m and n satisfying (i)-(v), let $t_1=(2n-m)/3$ and $t_2=(2m-n)/3$. Then t_1 and t_2 are integers such that $0 \leqq t_1 \leqq m$ and $0 \leqq t_2 \leqq n$. Hence, $t_1+2t_2=m$ and $2t_1+t_2=n$. Using t_1 vertices in V_1 and $2t_1$ vertices in V_2, consider t_1 S_3's whose endvertices are in V_2. Using the remaining $2t_2$ vertices in V_1 and the remaining t_2 vertices in V_2, consider t_2 S_3's whose endvertices are in V_1. Then these t_1+t_2 S_3's are edge-disjoint and they form an S_3-factor of $K^*_{m,n}$. □

Corollary 2. $K^*_{n,n}$ has an S_3-factor if and only if $n \equiv 0 \pmod 3$.

3. S_3-Factorization Algorithm of $K^*_{m,n}$

Notation 3. **r,t,b** : number of S_3-factors, number of S_3-components of each S_3-factor, and total number of S_3-components, respectively, in an S_3-factorization of $K^*_{m,n}$.

t_1 (t_2) : number of components whose centers are in V_1 (V_2), respectively, among t S_3-components of each S_3-factor.

$r_1(u)$ ($r_2(v)$) : number of components whose centers are all u (v) for any u (v) in V_1 (V_2), respectively, among b S_3-components.

$s_1(u)$ ($s_2(v)$) : number of components which have endvertex u (v) for any u (v) in V_1 (V_2), respectively, among b S_3-componets.

3.1. Necessary Condition of S_3-Factorization of $K^*_{m,n}$

Theorem 4. If $K^*_{m,n}$ has an S_3-factorization then $m=n \equiv 0 \pmod 6$.

Proof. Suppose that $K^*_{m,n}$ has an S_3-factorization. Then it holds that $b=mn$, $t=(m+n)/3$, $r=b/t=3mn/(m+n)$, $t_1=(2n-m)/3$, $t_2=(2m-n)/3$, $m \leqq 2n$ and $n \leqq 2m$. And it holds that $r_1(u)+s_1(u)=r$, $2r_1(u)=n$, $s_1(u)=n$, $r_2(v)+s_2(v)=r$, $2r_2(v)=m$, and $s_2(v)=m$. Therefore, $r_1(u)$ and $s_1(u)$ ($r_2(v)$ and $s_2(v)$) don't depend on u (v), respectively. Thus we have $r=r_1(u)+s_1(u)=3n/2$ and $r=r_2(v)+s_2(v)=3m/2$. Therefore, m=n is necessary. Moreover, when m=n, we have $b=n^2$, $t=2n/3$, $r=3n/2$, $t_1=t_2=n/3$, $r_1=r_2=n/2$,

$s_1 = s_2 = n$. Therefore, $n \equiv 0 \pmod 6$ is also necessary. □

3.2. Extension Theorem of S_3-Factorization of $K^*_{m,n}$

Theorem 5. If $K^*_{m,n}$ has an S_3-factorization, then $K^*_{sm,sn}$ has an S_3-factorization for every positive integer s.

Proof(by algorithm). 1: Let V_1, V_2 be the independent sets of $K^*_{sm,sn}$, where $|V_1| = sm$ and $|V_2| = sn$.

2: Divide V_1 and V_2 into s subsets of m and n vertices each, respectively.

3: Construct a new graph G with a vertex set consisting of the subsets which were just constructed.

4: In this graph, two vertices are symmetrically adjacent if and only if the subsets come from disjoint independent sets of $K^*_{sm,sn}$.

5: G is a complete bipartite symmetric digraph $K^*_{s,s}$.

6: Noting that the cardinality of each subset identified with a vertex set of G is m or n and that $K^*_{s,s}$ has a $K^*_{1,1}$-factorization, we see that the desired result is obtained.

3.3. Sufficient Condition of S_3-Factorization of $K^*_{m,n}$

Theorem 6. When $m=n \equiv 0 \pmod 6$, $K^*_{m,n}$ has an S_3-factorization.

Proof(by algorithm). 1: Put $m=n=6s$.

2: When s=1, we have the following lemma.

Lemma 7. $K^*_{6,6}$ has an S_3-factorization.

Proof. Let $V_1=\{1,2,3,4,5,6\}$ and $V_2=\{1',2',3',4',5',6'\}$. Let

F_1: $1' \leftarrow 1 \rightarrow 2'$ $3' \leftarrow 2 \rightarrow 4'$ $3 \leftarrow 6' \rightarrow 4$ $5 \leftarrow 5' \rightarrow 6$

F_2: $3' \leftarrow 1 \rightarrow 4'$ $5' \leftarrow 2 \rightarrow 6'$ $3 \leftarrow 1' \rightarrow 4$ $5 \leftarrow 2' \rightarrow 6$

F_3: $5' \leftarrow 1 \rightarrow 6'$ $1' \leftarrow 2 \rightarrow 2'$ $3 \leftarrow 3' \rightarrow 4$ $5 \leftarrow 4' \rightarrow 6$

F_4: $1' \leftarrow 3 \rightarrow 2'$ $3' \leftarrow 4 \rightarrow 4'$ $5 \leftarrow 6' \rightarrow 6$ $1 \leftarrow 5' \rightarrow 2$

F_5: $3' \leftarrow 3 \rightarrow 4'$ $5' \leftarrow 4 \rightarrow 6'$ $5 \leftarrow 1' \rightarrow 6$ $1 \leftarrow 2' \rightarrow 2$

F_6: $5' \leftarrow 3 \rightarrow 6'$ $1' \leftarrow 4 \rightarrow 2'$ $5 \leftarrow 3' \rightarrow 6$ $1 \leftarrow 4' \rightarrow 2$

F_7: $1' \leftarrow 5 \rightarrow 2'$ $3' \leftarrow 6 \rightarrow 4'$ $1 \leftarrow 6' \rightarrow 2$ $3 \leftarrow 5' \rightarrow 4$

F_8: $3' \leftarrow 5 \rightarrow 4'$ $5' \leftarrow 6 \rightarrow 6'$ $1 \leftarrow 1' \rightarrow 2$ $3 \leftarrow 2' \rightarrow 4$

F_9: $5' \leftarrow 5 \rightarrow 6'$ $1' \leftarrow 6 \rightarrow 2'$ $1 \leftarrow 3' \rightarrow 2$ $3 \leftarrow 4' \rightarrow 4$.

These comprise an S_3-factorization of $K^*_{6,6}$.

3: Applying Theorem 3, $K^*_{6s,6s}$ has an S_3-factorization for every positive integer s.

4: This completes the proof of Theorem 6. □

3.4. Conclusion

Theorem 8. $K^*_{m,n}$ has an S_3-factorization if and only if $m=n \equiv 0 \pmod 6$.

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