STAR-FACTORIZATION ALGORITHMS OF COMPLETE GRAPHS BY RBIB DESIGNS

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1. Introduction

Let S_k be a ***star*** with k vertices. Let K_k and K_v be a ***complete graph*** with k vertices and v vertices, respectively. Let K(m;k) be a ***complete multipartite graph*** with m partite sets of k vertices each.

Let G and H be graphs. A spanning subgraph F of G is called an H-***factor*** if and only if each component of F is isomorphic to H. If G is expressible as an edge-disjoint sum of H-factors, then this sum is called an H-***factorization*** of G.

2. Common Transversal Sets

We use common transversal sets on S_k-factorization of K_v and K(m;k).

Consider a set N of v elements, where v=kt. Divide N into t subsets $A_1, A_2, \dots, A_t$ so that they are mutually disjoint subsets of same size k. And divide N into another t subsets $B_1, B_2, \dots, B_t$ so that they are mutually disjoint subsets of same size k. Let T be a t-element subsets of N. Then T ia called a ***common transversal set*** of $\{A_1, A_2, \dots, A_t\}$ and $\{B_1, B_2, \dots, B_t\}$ when it holds that $|T \cap A_j| = |T \cap B_j| = 1$, $1 \leq j \leq t$.

Lemma 1. *Let $\{A_1, A_2, \dots, A_t\}$ be a mutually disjoint partition of N and $\{B_1, B_2, \dots, B_t\}$ be another mutually disjoint partition of N. Then there exists a common transversal set T of $\{A_1, A_2, \dots, A_t\}$ and $\{B_1, B_2, \dots, B_t\}$.*

3. S_k-Factorization Algorithms of K_v and K(m;k)

For $k \geqq 3$, we have the following:

Lemma 2. *An edge-disjoint sum of two K_k-factors of K_v can be factorized into k S_k-factors.*

Proof(by algorithm). 1: Let F_1 and F_2 be edge-disjoint K_k-factors of K_v. And let G be a sum of F_1 and F_2. Then $V(F_1)=V(F_2)=V(G)=V(K_v)$.

2: Put $F_1 = K_k^{(1)} \cup K_k^{(2)} \cup \dots \cup K_k^{(t)}$ and $F_2 = K_k^{(t+1)} \cup K_k^{(t+2)} \cup \dots \cup K_k^{(2t)}$.
And let $A_j = V(K_k^{(j)})$ and $B_j = V(K_k^{(t+j)})$, $1 \leqq j \leqq t$.
Then $\{A_1, A_2, \dots, A_t\}$ is a mutually disjoint partition of V(G) and $\{B_1, B_2, \dots, B_t\}$ is another mutually disjoint partition of V(G).

3: Let T_1 be a common transversal set of $\{A_1, A_2, \dots, A_t\}$ and $\{B_1, B_2, \dots, B_t\}$.
Let T_j be a common transversal set of $\{A_1-T, A_2-T, \dots, A_t-T\}$ and $\{B_1-T, B_2-T, \dots, B_t-T\}$, where $T = T_1 + T_2 + \dots + T_{j-1}$ $(2 \leqq j \leqq k)$.

4: Consider 2k-2 subgraphs G_j of G such as $G_1 = F_1$, $G_2 = F_2$, $G_j = G_{j-2} - T_{j-2}$ $(3 \leqq j \leqq 2k-2)$, where $T_{k+i}=T_{k-i+1}$ $(1 \leq i \leq k-2)$.

5: Consider 2k−2 subgraphs H_j of G such as $H_j=G_j-E(G_j-T_j)$ $(1 \leq j \leq 2k-4)$, $H_{2k-3}=G_{2k-3}$, $H_{2k-2}=G_{2k-2}$.

6: Note that every component of G_j is a complete graph with (2k−j+1)/2 vertices (j:odd) or (2k−j+2)/2 vertices (j:even) and that every component of H_j is a star with (2k−j+1)/2 vertices (j:odd) or (2k−j+2)/2 vertices (j:even).

7: Then we can construct k edge-disjoint S_k-factors $F_1', F_2', F_3, \ldots, F_k$ of G as follows:

$$F_1'=H_1,\ F_2'=H_2,\ F_j=H_j \cup H_{2k-j+1}\ (3 \leq j \leq k).$$

8: Therefore, it holds that $G=F_1'(+)F_2'(+)F_3(+)\ldots(+)F_k$, which is an S_k-factorization, where the symbol (+) is used to denote the sum of factors.□

As a resolvable BIBD(v,b,r,k, λ =1) is just a K_k-factorization of K_v, we have the following lemmas.

Lemma 3. *If there exists a resolvable BIBD(v,b,r,k, λ =1) (r:even), then K_v has an S_k-factorization.*

Proof. Let $F_1, F_2, \ldots, F_r$ be edge-disjoint K_k-factors of K_v. And let G_i be a sum of F_{2i-1} and F_{2i} $(1 \leq i \leq r/2)$. Then from Lemma 2, G_i can be factorized into k S_k-factors. Therefore, K_v has an S_k-factorization.□

Lemma 4. *If there exists a resolvable BIBD(v,b,r,k, λ =1) (r:odd, m=v/k), then K(m;k) has an S_k-factorization.*

Proof. Let $F_1, F_2, \ldots, F_r$ be edge-disjoint K_k-factors of K_v. Then we can write as follows:

$$K_v=F_1(+)F_2(+)\ldots(+)F_r=F_1(+)K(m;k),\text{ where } K(m;k)=F_2(+)\ldots(+)F_r.$$

Let G_i be a sum of F_{2i} nd F_{2i+1} $(1 \leq i \leq (r-1)/2)$. Then from Lemma 2, G_i can be factorized into k S_k-factors. Therefore, K(m;k) has an S_k-factorization.□

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