Regular Paper

Cellular Automata on Groups with Asymptotic Boundary Conditions

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Cellular automata on groups with asymptotic boundary conditions are studied. The main results are non-Euclidean extensions of Maruoka and Kimura's results on injectivity and surjectivity. They introduced the notions of weak injectivity/surjectivity and strong injectivity/surjectivity, and showed a hierarchical structure among these properties. The Garden of Eden (GOE) property and the periodic construction technique are used to extend their results. Groups that are residually finite and of non-exponential growth are shown to form a good class for non-Euclidean extensions of classical results.

1. Introduction

Cellular automata on Euclidean cell spaces have been investigated by many researchers. A successful attempt at non-Euclidean extension was made by Machì and Mignosi[4]. They proved the Garden of Eden (GOE) theorem, an extension of Moore-Myhill's theorem, where cell spaces are taken from Cayley graphs of groups with non-exponential growth. An alternative proof of the GOE theorem was given by the author; though limited to the Heisenberg groups, it has the advantage of explicitly constructing Moore-Myhill pseudo-tilings[15]. Recent progress and classical results including Maruoka and Kimura's results mentioned below are concisely summarized in Garzon's book[2].

This paper is another attempt at non-Euclidean extension. We first briefly review the classical results that we will focus on.

In the 1960s and 1970s, injectivity and surjectivity of parallel maps were investigated in connection with Moore-Myhill's theorem[1),12),13)], known as the GOE theorem, which established the equivalence of surjectivity and local injectivity (the non-existence of mutually erasable patterns). Following these earlier studies, Maruoka and Kimura introduced variants of the notions of injectivity and surjectivity—the notions of weak injectivity/surjectivity and strong injectivity/surjectivity—and showed various relations among these properties[9]. Maruoka and Kimura[10] finally obtained a remarkable result concerning the hierarchy among properties of injectivity and surjectivity, which we will call *Maruoka-Kimura's hierarchy*, or the *M-K hierarchy* for short. See **Fig. 1**, where (1) crossed arrows mean there exists a counterexample that disproves the implication, (2) properties enclosed in the same square are all equivalent, and (3) the terms *totally injective/surjective* are used to contrast the usual surjectivity and injectivity with their other versions. All the above studies deepened our understanding of injectivity and surjectivity of parallel maps.

Can we reproduce the beautiful hierarchy in a non-Euclidean framework? Attempts must face the following problem. Maruoka and Kimura's discussions[9),10)] depend heavily on the notions of *balancedness* and *hardness* of parallel maps and the following characterizations[8]:

(1) A parallel map is surjective if and only if it is balanced.

(2) A parallel map is injective if and only if it is hard.

Balancedness and hardness are determined by counting the number of pre-images of local configurations on the squares. While the notion of hardness and the characterization of injectivity can be easily extended to non-Euclidean cellular automata, an appropriate definition of

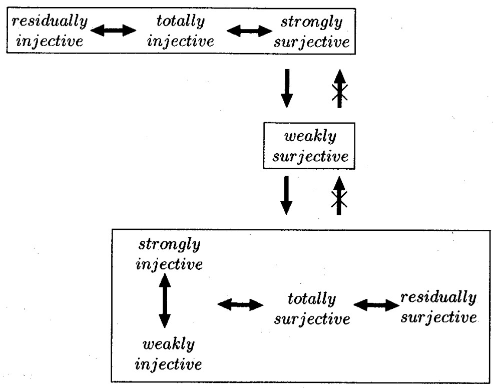

Fig. 1 The M-K hierarchy.

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balancedness for them has not been found yet.

Because of this deficiency, we have to abandon any attempts at a complete reproduction of their results in a non-Euclidean framework. The goal of this paper is to show modified hierarchies, where some conditions are imposed on the group that generates the tessellation. The *GOE property*, *residual finiteness*, and their combination are considered.

The organization of this paper is as follows. The main results are formulated and proved in Sections 4, 6, and 8. In Section 2, we describe the topology of the configuration spaces of cellular automata on Cayley graphs and define the group actions on the configuration spaces. In Section 3, we describe basic notions in Maruoka-Kimura's theory in a form adapted for our purpose. In Section 4, we state and prove the first main theorem on the M-K hierarchy under a general condition. In Section 5, we collect definitions and results on the GOE property. In Section 6, we state and prove the second main theorem on the M-K hierarchy with the GOE property. In Section 7, we briefly review the periodic construction technique recently introduced by the author. In Section 8, we state the third main theorem on the M-K hierarchy with residual finiteness. The proof is based on the periodic construction. In Section 9, assuming both the GOE property and residual finiteness, we show the fourth main theorem, which nearly reproduces the original M-K hierarchy.

2. Cellular Automata on Groups

In this section, we summarize the basic definitions and properties of cellular automata on groups.

Let G be a group. The *Cayley graph of G with respect to a subset N' of G* is a directed graph with vertex set G and edge set $E = G \times N'$, and incidence functions $\iota(g,h) = g$, $\tau(g,h) = gh$ for all $(g,h) \in G \times N'$. We denote this graph by $\Gamma(G, N')$. To attain meaningful results, we assume throughout this paper that G is a finitely generated infinite group and N' is a finite set of generators.

Let G be as above. We can define a metric ρ on G derived from the path metric on the Cayley graph $\Gamma(G, N')$. We denote by e the identity element in G and by B_n a ball of radius n with center e, namely $B_n = \{g \in G \mid \rho(g,e) \leq n\}$.

Let Q be a finite set that we call the set of *states*. A *local map* is a map $\sigma : Q^N \to Q$, where N is a finite subset of G. We call N the *support* of σ. We always assume that $e \in N$ and $N^{-1} = N$. An element of Q^G, that is, a map $G \to Q$ is called a *configuration*. Instead of Q^G we write C for short. We define the *shift* $s_g : C \to C$ induced by $g \in G$ as follows. For any $x \in C$, we define $s_g(x) \in C$ by $[s_g(x)](h) = x(g^{-1}h)$ for all $h \in G$. We define the *parallel map* $T_\sigma : C \to C$ induced by σ as $(T_\sigma(x))(g) = \sigma(s_g^{-1}(x)|_N)$ for all $x \in C$, $g \in G$, where N is the support of σ. The pair (C, T_σ) is a discrete dynamical system and is called a *cellular automaton*.

Remark 1. If we enumerate the elements of N with indices as $h_1, h_2, \ldots, h_n$, we may write a local map σ as a function of n variables $\sigma(q_1, q_2, \ldots, q_n)$. Then, the above definition of the parallel map is rewritten as

$$(T_\sigma(x))(g) = \sigma(x(gh_1), x(gh_2), \ldots, x(gh_n)) \quad \text{for all } x \in C,\ g \in G.$$

Furthermore, if G is Abelian, it is common to adopt the additive notation "$g + h$." Then we obtain a familiar expression $(T_\sigma(x))(g) = \sigma(x(g+h_1), \ldots, x(g+h_n))$ for Euclidean tessellation automata. These formulations are intuitive, but they have the disadvantage that the local map and the enumeration of the neighborhood cells must be separately specified.

An element of Q^A, that is, a map $A \to Q$, is called a *local configuration over A*, where $A \subset G$. We write Q^A as $C(A)$. We define the *local shift* $s_g : C(A) \to C(gA)$ induced by $g \in G$ as follows. For any $x \in C(A)$, we define $s_g(x) \in C(gA)$ by $[s_g(x)](h) = x(g^{-1}h)$ for all $h \in gA$. Since there can be no confusion, we use the same notation for the shift $C \to C$ and for the local shift $C(A) \to C(gA)$. Further, we say simply "shift" instead of "local shift."

Let A and B be two subsets of the group G. We denote by AB the subset of G defined by $\{ab \in G \mid a \in A, \quad b \in B\}$.

Let N be the support of a local function σ. If $AN \subset B$, we can define the local parallel map $T_{\sigma,B,A} : C(B) \to C(A)$ in an obvious way: $(T_{\sigma,B,A}(x))(g) = \sigma(s_g^{-1}(x)|_N)$ for all $x \in C(B)$, $g \in A$.

Let A and B be two disjoint subsets of G and let $x \in C(A)$ and $y \in C(B)$. We define the concatenation of x and y as a local configuration on $A \cup B$ such that its restriction on A coincides with x and its restriction on B coincides with y, and denote it by $x \sqcup y \in C(A \cup B)$. The condition above may be written as $(x \sqcup y)|_A = x$ and $(x \sqcup y)|_B = y$. Let $\{A_k \mid k \in K\}$ be a mutually disjoint family of subsets of G. Let $x_k \in C(A_k)$. We denote and define the concatenation of local configurations x_k's by $\sqcup_{k\in K} x_k \in C(\cup_{k\in K} A_k)$ where $(\sqcup_{k\in K} x_k)|_{A_k} = x_k$ for all $k \in K$.

We give the product topology to the space $C = Q^G$, where Q is endowed with discrete topology. To be more specific, we introduce cylinder sets as follows. Let A be a finite subset of G and $K \subset C(A)$. We define the cylinder set of K as $\{x \in C \mid x|_A \in K\}$ and denote it by $\mathrm{Cyl}\,[K]$. If K consists of only one element $y \in C(A)$, then we write $\mathrm{Cyl}\,[y]$ instead of $\mathrm{Cyl}\,[\{y\}]$, for brevity. The set of all such cylinders forms the basis of the product topology. The well-known Tichonov's theorem ensures that C is compact. It is clear that for any $g \in G$ the shift map $s_g : C \to C$ is a homeomorphism and for any local map σ the parallel map T_σ is continuous.

We say that a map $p : C \to C$ is *cellular* if there exists a set $N \subset G$ and a local map $\sigma : Q^N \to Q$ such that $p = T_\sigma$. The following lemma is due to Richardson[14]. See also Garzon's book[2].

Lemma 1. A map $p : C \to C$ is cellular if and only if it is continuous and commutes with all the shifts s_g $(g \in G)$.

The following lemma can be found in any standard textbook of general topology[3].

Lemma 2. If X and Y are compact spaces and $f : X \to Y$ is a surjective continuous map, then f is an open map, namely, it maps every open set in X onto an open set in Y. Especially when f is bijective, f^{-1} is continuous.

3. Asymptotic Boundary Conditions

The notions of strong/weak injectivity and strong/weak surjectivity were introduced by Maruoka and Kimura[9]. With these notions, they found various intermediate properties of parallel maps that come between surjectivity and injectivity.

An equivalence relation $\asymp$ in C is defined as follows. Let x and y be two configurations. We say that x and y are *asymptotically equivalent* and write $x \asymp y$ if $x(g) = y(g)$ for all but a finite number of $g \in G$. We denote by C_x the equivalence class that contains x. The equivalence class C_x may be seen as the set of configurations with a given asymptotic boundary condition at "infinity." We denote by $C/\!\asymp$ the quotient space, that is, the set of all asymptotic equivalence classes. For any $x \asymp y$, we have $T_\sigma(x) \asymp T_\sigma(y)$. This means that T_σ maps C_x into $C_{T_\sigma(x)}$ for any $x \in C$ and thereby induces the quotient map $T_\sigma/\!\asymp : C/\!\asymp \to C/\!\asymp$.

The following lemma is obvious:

Lemma 3. For each $x \in C$, C_x is dense in C.

Definition 1. We say that T_σ is *strongly injective* if $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$ is injective for all $x \in C$. We say that T_σ is *strongly surjective* if $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$ is surjective for all $x \in C$. We say that T_σ is *weakly injective* if there exists $x \in C$ such that $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$ is injective. We say that T_σ is *weakly surjective* if there exists $x \in C$ such that $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$ is surjective. We say that T_σ is *residually injective* if $T_\sigma/\!\asymp : C/\!\asymp \to C/\!\asymp$ is injective. We say that T_σ is *residually surjective* if $T_\sigma/\!\asymp : C/\!\asymp \to C/\!\asymp$ is surjective.

When we want to contrast the usual surjectivity and injectivity with their other versions, we say that T_σ is *totally injective/surjective* if $T_\sigma : C \to C$ is injective/surjective.

4. The M-K Hierarchy in General

We state and prove the first main theorem concerning the M-K hierarchy in general. As mentioned earlier, G is assumed throughout this paper to be a finitely generated infinite group.

Theorem 1. For any local map σ, relations among properties of T_σ are summarized in the diagrams in **Fig. 2**.

Figure 2 consists of three disconnected components separated by dashed lines. Notice that residual injectivity is isolated. Though such a component could be omitted, we leave it among other components for ease of comparison with Fig. 1.

First, we prove the component of Fig. 2 that relates the three versions of injectivity.

The lemma below directly follows from the definitions.

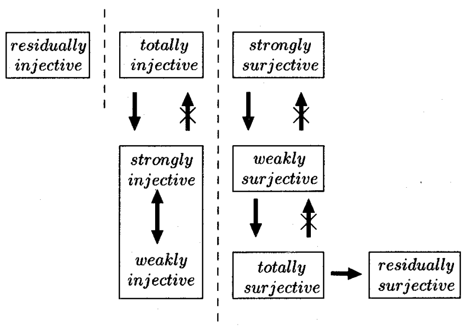

Fig. 2 The M-K hierarchy in general.

Lemma 4. (i) If a parallel map T_σ is totally injective, then it is strongly injective.
(ii) If a parallel map T_σ is strongly injective, then it is weakly injective.

Remark 2. An example of a parallel map that is strongly injective and not totally injective will be obtained from either Remark 3 or Remark 4.

The non-trivial part of the component is the equivalence of strong injectivity and weak injectivity. This is established by the next lemma.

Lemma 5. If a parallel map T_σ is weakly injective, then it is strongly injective.

Proof. We assume that $T_\sigma|_{C_{x_0}} : C_{x_0} \to C_{T_\sigma(x_0)}$ is injective and that there exist two configurations x_1 and x_2 such that $x_1 \neq x_2$, $x_1 \asymp x_2$, and $T_\sigma(x_1) = T_\sigma(x_2)$. Namely, we assume $T_\sigma|_{C_{x_1}} : C_{x_1} \to C_{T_\sigma(x_1)}$ is not injective. We can find an integer n such that $x_1|_{G-B_n} = x_2|_{G-B_n}$ and $x_1|_{B_n} \neq x_2|_{B_n}$. Let y_1 and y_2 be two configurations defined by

$$y_1 = x_0|_{G-B_{n+2}} \sqcup x_1|_{B_{n+2}},$$
$$y_2 = x_0|_{G-B_{n+2}} \sqcup x_2|_{B_{n+2}},$$

as in **Fig. 3**. Then, y_1 and y_2 are different configurations in C_{x_0} and yet we can show that $T_\sigma(y_1) = T_\sigma(y_2)$ as follows:

(i) $T_\sigma(y_1)|_{G-B_{n+1}} = T_\sigma(y_2)|_{G-B_{n+1}}$,
since $y_1|_{G-B_n} = y_2|_{G-B_n}$
and $(G - B_{n+1})N \subset G - B_n$.

(ii) $T_\sigma(y_1)|_{B_{n+1}} = T_\sigma(y_2)|_{B_{n+1}}$,
since $y_1|_{B_{n+2}} = x_1|_{B_{n+2}}$,
$y_2|_{B_{n+2}} = x_2|_{B_{n+2}}$,
$B_{n+1}N = B_{n+2}$,
and $T_{\sigma,B_{n+1}N,B_{n+1}}(x_1|_{B_{n+1}N})$
$= T_{\sigma,B_{n+1}N,B_{n+1}}(x_2|_{B_{n+1}N})$.

Thus, we see $T_\sigma(y_1) = T_\sigma(y_2)$. This contradicts the assumption that $T_\sigma|_{C_{x_0}}$ is injective. Now we can conclude that weak injectivity implies strong injectivity. □

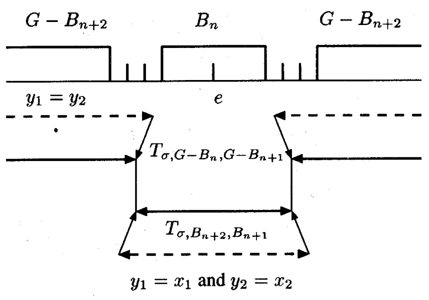

Fig. 3 y_1 and y_2.

Next, we prove the rightmost component in Fig. 2 that relates the four versions of surjectivity. The proof consists of two lemmas.

The following lemma is trivial from the definition.

Lemma 6. (i) If a parallel map T_σ is strongly surjective, then it is weakly surjective.
(ii) If a parallel map T_σ is totally surjective, then it is residually surjective.

Remark 3. Maruoka and Kimura gave an example of a parallel map that is weakly surjective but not strongly surjective (in the proof of Theorem 22 [9]).

The next lemma is purely topological.

Lemma 7. If a parallel map T_σ is weakly surjective, then it is totally surjective.

Proof. Assume that $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$ is surjective for some $x \in C$. We will prove the surjectivity of $T_\sigma : C \to C$ by deriving the equation chain

$$T_\sigma(C) = T_\sigma(\overline{C_x}) = \overline{T_\sigma(C_x)} = \overline{C_{T_\sigma(x)}} = C,$$

where $\overline{X}$ means the closure of a set X in C.

The first and the last equalities follow from the fact that C_x and $C_{T_\sigma(x)}$ are dense in C, that is, $\overline{C_x} = \overline{C_{T_\sigma(x)}} = C$. The third equality follows from the assumption $T_\sigma(C_x) = C_{T_\sigma(x)}$. The second equality is derived from the following argument. Since T_σ is continuous, we have $T_\sigma(\overline{C_x}) \subset \overline{T_\sigma(C_x)}$. To show the inverse inclusion, recall that C is compact. In general, any continuous map on a compact space is closed, that is, the image of a closed set is also closed. Thus we have $\overline{T_\sigma(\overline{C_x})} = T_\sigma(\overline{C_x})$. We also have an obvious inclusion $\overline{T_\sigma(C_x)} \subset \overline{T_\sigma(\overline{C_x})}$. This establishes the second equality and completes the proof. □

Remark 4. Maruoka and Kimura gave an example of a parallel map that is strongly injective but not weakly surjective, (in the proof of Theorem 21 [9]). In their context, the example was also shown to be totally surjective. Therefore, total surjectivity does not imply weak surjectivity.

Remark 5. Let $f : X \to Y$ be a continuous map. For any compact set $U \subset X$, the image $f(U)$ is also compact. If X is a compact space, any closed set is compact. Applying these two facts to our case, we know that $T_\sigma : C \to C$ is a closed map (see any standard textbook on general topology [3]).

Lemmas 4–7 complete the proof of Theorem 1.

5. The GOE Theorem

We introduce Machì and Mignosi's Garden of Eden (GOE) theorem, which will be used as an essential element in the lemmas that lead to the second main theorem in Section 6.

Let G be a finitely generated group with a set of generators N'. Recall that we can define a metric on G derived from the path metric on the Cayley graph $\Gamma(G, N')$, and let B_n be a ball of radius n with center e. The *growth function* of a finitely generated group G is defined by $\gamma(n) = |B_n|$. If $\liminf \gamma(n)/\gamma(n-1) = 1$, we say G is of *non-exponential growth*. This property does not depend on the choice of a generator set, which justifies the above definition [11].

Whenever a local map σ is fixed in the context, the ball B_n is to be defined by the metric on G derived from the path metric on $\Gamma(G, N - \{e\})$ as above, where N is the support of σ.

Let σ be a local map and N be its support. Since we assumed that $e \in N$, we have $N^2 = NN \supset N$. We say that x and y are *mutually erasable* and that T_σ is *erasing* if there exist a finite set $A \subset G$ and two local configurations $x, y \in C(AN^2)$ such that

$$x|_{AN^2-A} = y|_{AN^2-A}, \quad x|_A \neq y|_A,$$

and

$$T_{\sigma,AN^2,AN}(x) = T_{\sigma,AN^2,AN}(y).$$

Notice that the existence of mutually erasable local configurations implies that T_σ is not injective.

Definition 2. We say that a group G has the *GOE* property if the following condition is satisfied:

Condition: A parallel map T_σ is surjective if and only if it is not erasing.

Lemma 8 (The Garden of Eden Theorem [4]). If G is a group of non-exponential growth, then it has the GOE property.

Remark 6. Machì and Mignosi [4] gave an example of exponential growth that does not have the GOE property. Examples of groups of non-exponential growth are given in the author's recent paper [15].

Combining Lemmas 1 and 8, we obtain the following:

Lemma 9. Assume that G has the GOE property. If a parallel map T_σ is injective, then it is also surjective and T_σ^{-1} is cellular.

Proof. Injectivity of T_σ implies that T_σ is not erasing. Therefore, by the GOE property, any injective parallel map T_σ is bijective. Recall that the configuration space C is compact. From Lemma 2, the inverse map T_σ^{-1} is continuous. Since T_σ^{-1} commutes with all the shifts, we conclude that T_σ^{-1} is cellular. □

6. The M-K Hierarchy with the GOE Property

We state and prove the second main theorem concerning the M-K hierarchy with the GOE property.

Theorem 2. Assume that G has the GOE property. For any local map σ, relations among properties of T_σ are summarized as in **Fig. 4**.

The diagram in Fig. 4 is obtained from the diagram in Fig. 2 by adding two unidirectional and one bidirectional arrows. Since Theorem 1 was already proved under the more general condition, we only discuss the additional arrows.

Lemma 10. Assume that G has the GOE property. If a parallel map T_σ is totally injective, then T_σ is strongly surjective.

Proof. If T_σ is totally injective, then by Lemma 9, T_σ^{-1} exists and is cellular. Let $x \in C$ be an arbitrary configuration. We must show that $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$ is surjective. Let y be any element in $C_{T_\sigma(x)}$. Since T_σ^{-1} is cellular, we have $T_\sigma^{-1}(y) \asymp T_\sigma^{-1}(T_\sigma(x)) = x$, which means that $T_\sigma^{-1}(y) \in C_x$. Thus, we know that $T_\sigma|_{C_x}$ is surjective. Since x was taken arbitrarily, we conclude that T_σ is strongly surjective. □

Lemma 11. Assume that G has the GOE property. If a parallel map T_σ is totally surjective, then it is strongly injective.

Proof. We prove the contrapositive. Assume that $T_\sigma|_{C_x}$ is not injective for some $x \in C$. Let x_1 and x_2 be two distinct configurations in C_x

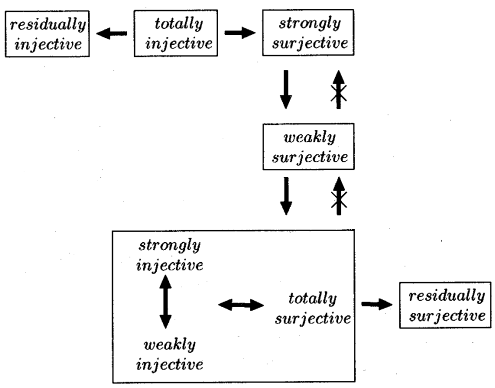

Fig. 4 The M-K hierarchy with the GOE property.

such that $T_\sigma(x_1) = T_\sigma(x_2)$. We can find a finite subset $B \subset G$ such that

$$x_1|_{G-B} = x_2|_{G-B} = x|_{G-B}$$

and

$$x_1|_B \neq x_2|_B.$$

This means that $x_1|_{BN^2}$ and $x_2|_{BN^2}$ are mutually erasable. From the GOE property, we deduce that T_σ in not totally surjective. □

Lemma 12. Assume that G has the GOE property. If a parallel map T_σ is weakly injective, then it is totally surjective.

Proof. We prove the contrapositive. Assume that T_σ is not totally surjective. From the GOE property, there are mutually erasable local configurations $x_1, x_2 \in C(BN^2)$, where B is a finite subset of G. Let $x \in C$ be an arbitrary configuration. If we extend x_1 and x_2 in such a way that $x_1|_{G-BN^2} = x_2|_{G-BN^2} = x|_{G-BN^2}$, then we see that $T_\sigma|_{C_x}$ is not injective as in the proof of Lemma 5. Since x was taken arbitrarily, T_σ cannot be weakly injective. □

Lemma 13. Assume that G has the GOE property. If a parallel map T_σ is totally injective, then T_σ is residually injective.

Proof. Let us assume that T_σ is totally injective. Then, by the GOE property and Lemma 9, T_σ is bijective and T_σ^{-1} is cellular. Let x_1 and x_2 be two configurations such that $T_\sigma(x_1) \asymp T_\sigma(x_2)$. Since T_σ^{-1} is cellular, we have $T_\sigma^{-1}(T_\sigma(x_1)) \asymp T_\sigma^{-1}(T_\sigma(x_2))$ and consequently $x_1 \asymp x_2$. This shows that $T_\sigma/\asymp$ is injective. □

Lemmas 10–13 together with Theorem 1 complete the proof of Theorem 2.

7. Periodic Constructions

A periodic construction out of a local configuration over a finite set is conceptually depicted in **Fig. 5**, where the shaded region represents the given local configuration. We will use later such a construction in non-Euclidean cell spaces.

However, periodic constructions are not possible for all groups. Residually finite groups are shown to have good properties for the study of $T_\sigma/\asymp$ in Section 8. We give definitions and state some useful properties for later use.

We say that a group G is *residually finite* if the intersection of all normal subgroups with finite index is the trivial subgroup $\{e\}$. The following lemma is directly deduced from the definition. See chapter 2 of Magnus, Karrass, and Solitar's book[6].

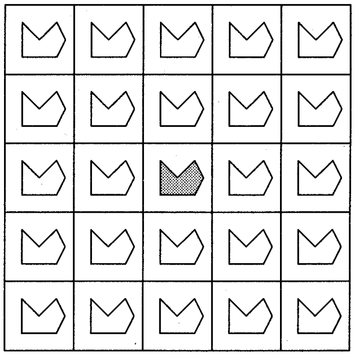

Fig. 5 Periodic construction.

Lemma 14. (i) A group G is residually finite if and only if for every $g \in G - \{e\}$ there is a subgroup H of finite index with $g \notin H$.
(ii) A group G is residually finite if and only if for every finite subset B of $G - \{e\}$ there is a subgroup H of finite index with $H \cap B = \emptyset$.

Note that, in the statements above, we may replace "subgroup" by "normal subgroup."

It can be easily seen that lattices in Euclidean spaces are residually finite. Free groups, Fuchsian groups are known to be residually finite. Further, in general, finitely generated subgroups of general linear groups $GL(n, R)$, where R is any commutative field, are residually finite[5),7)].

Let x be a configuration. The *period* of x is a subgroup of G defined by $\{g \in G \mid s_g(x) = x\}$. We denote it by $\omega(x)$. If $\omega(x)$ is of finite index, we say that x is *periodic* or *cofinite*.

Let H be a subgroup of G. We denote by $H\backslash G$ the right coset decomposition. A *fundamental transversal* A of H is a complete set of representatives of $H\backslash G$, that is, a subset of G such that $HA = G$ and $Ha_1 \neq Ha_2$ for any distinct $a_1, a_2 \in A$. We have the natural projection $\pi : G \to H\backslash G$. If we identify the fundamental transversal A with $H\backslash G$, we have a projection $\pi_{H,A} : G \to A$. Let $x \in C(A)$. The pullback $\pi^*_{H,A} : C(A) \to C$ is defined by $(\pi^*_{H,A}x)(ha) = x(a)$ for all $h \in H$, $a \in A$. The pullback can also be written as $\pi^*_{H,A}x = \sqcup_{h \in H} s_h(x)$. Notice that $\omega(\pi^*_{H,A}x) \geq H$, where $X \geq Y$ means that Y is a subgroup of X.

We have the following lemma:

Lemma 15. Assume that G is residually finite. Given a local configuration x over a finite set B, we can construct a periodic configuration $\tilde{x}$ such that $\tilde{x}|_B = x$.

Proof. Since G is residually finite, there is a subgroup H of finite index such that $H \cap (B \cup BB^{-1}) = \{e\}$. This means that we can choose a fundamental transversal A that contains B. By using the same notation as above, we find that $\tilde{x} = \pi^*_{H,A}x$ is a periodic configuration with the desired property. □

Remark 7. In the above proof, the condition $H \cap (B \cup BB^{-1}) = \{e\}$ cannot be replaced with a simpler condition $H \cap B = \{e\}$. Under the latter condition, we cannot eliminate the possibility that $Hb_1 = Hb_2$ for $b_1, b_2 \in B$ with $b_1 \neq b_2$.

8. The M-K Hierarchy with Residual Finiteness

In this section, we state and prove the third main theorem concerning the M-K hierarchy with residual finiteness.

Theorem 3. Assume that G is residually finite. For any local map σ, relations among the properties of T_σ are summarized as in **Fig. 6**.

The following two lemmas together with Theorem 1 complete the proof of Theorem 3.

Lemma 16. Assume that G is residually finite. If a parallel map T_σ is residually injective, then T_σ is totally injective.

Proof. Let us assume that T_σ is not injective while $T_\sigma/\asymp$ is injective. Then, there are two configurations, $x_1 \neq x_2$, and a ball $B_n \subset G$, such that $T_\sigma(x_1) = T_\sigma(x_2)$ and $x_1|_{G-B_n} = x_2|_{G-B_n}$. Since G is residually finite, from Lemma 15, there are periodic configurations $\tilde{x}_1 = \pi^*_{H,A}(x_1|_A)$ and $\tilde{x}_2 = \pi^*_{H,A}(x_2|_A)$, where H is a subgroup of G with finite index and a fundamental transversal $A \supset B_{n+2}$. From these constructions we have $\tilde{x}_1(ha) = x_1(a)$ and $\tilde{x}_2(ha) = x_2(a)$ for $h \in H$, $a \in A$. Thus we can clearly see that $\tilde{x}_1 \not\asymp \tilde{x}_2$. We will show that $T_\sigma(\tilde{x}_1) = T_\sigma(\tilde{x}_2)$, which contradicts the injectivity of $T_\sigma/\asymp$.

First, we prove that

$$T_\sigma(\tilde{x}_1)|_{HB_{n+1}} = T_\sigma(\tilde{x}_2)|_{HB_{n+1}}$$

by deriving the equations

$$\begin{aligned}(T_\sigma(\tilde{x}_1))(ha) &= (T_\sigma(x_1))(a)\\ &= (T_\sigma(x_2))(a) = (T_\sigma(\tilde{x}_2))(ha)\end{aligned}$$

for $h \in H$, $a \in B_{n+1}$. The second equality follows from the assumption that $T_\sigma(x_1) = T_\sigma(x_2)$. The first and the third equalities follow from the condition that $B_{n+1}N = B_{n+2} \subset A$.

Next, we prove that

$$T_\sigma(\tilde{x}_1)|_{H(A-B_{n+1})} = T_\sigma(\tilde{x}_2)|_{H(A-B_{n+1})}.$$

Let $g \in H(A - B_{n+1})$. If we notice that $\rho(g, HB_n) \geq 2$, we have $\rho(gN, HB_n) \geq 1$, which indicates that $H(A - B_{n+1})N \subset G - HB_n$. Since $\tilde{x}_1|_{G-HB_n} = \tilde{x}_2|_{G-HB_n}$, we have $\tilde{x}_1|_{H(A-B_{n+1})N} = \tilde{x}_2|_{H(A-B_{n+1})N}$, and consequently $T_\sigma(\tilde{x}_1)|_{H(A-B_{n+1})} = T_\sigma(\tilde{x}_2)|_{H(A-B_{n+1})}$.

To sum up, we have $\tilde{x}_1 \not\asymp \tilde{x}_2$ and $T_\sigma(\tilde{x}_1) = T_\sigma(\tilde{x}_2)$, which means that $T_\sigma/\asymp$ is non-injective. Now, we conclude that if $T_\sigma/\asymp$ is injective then T_σ is injective. □

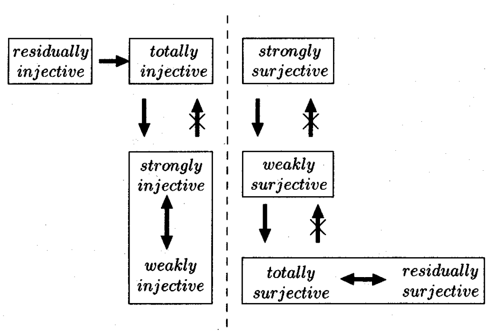

Fig. 6 The M-K hierarchy with residual finiteness.

Lemma 17. Assume that G is residually finite. If a parallel map T_σ is residually surjective, then T_σ is totally surjective.

Proof. Assume that $T_\sigma/\asymp$ is surjective. Let $K_0 \subset K_1 \subset K_2 \cdots$ be an ascending sequence of finite subsets in G with $\cup_{i\geq 0} K_i = G$. Let $y \in C$ be arbitrarily given. We define P_i to be a subset of C as

$$\begin{aligned}P_i &= T_\sigma^{-1}(\,\mathrm{Cyl}\,[y|_{K_i}])\\ &= \{x \in C \mid T_\sigma(x)|_{K_i} = y|_{K_i}\}.\end{aligned}$$

Since T_σ is continuous and any cylinder sets are closed, we see that P_i's are all closed. We immediately see that $P_0 \supset P_1 \supset P_2 \supset \ldots$. If we are able to show that all P_i's are nonempty, then, from the compactness of C, we have $\cap_{i\geq 0} P_i \neq \emptyset$, from which we can deduce that $T_\sigma^{-1}(y) \neq \emptyset$. Since y was taken arbitrarily, this shows the surjectivity of T_σ.

Now, it remains to show that all P_i's are nonempty. Since G is residually finite, we have a subgroup H_i with finite index such that K_i is included in a fundamental transversal A_i of $H_i \backslash G$. Let y be an arbitrary configuration, as above. Let $y_i = \pi^*_{H_i,A_i}(y|_{A_i}) = \sqcup_{h\in H_i} s_h(y|_{A_i})$. Then, from the surjectivity of $T_\sigma/\asymp$, there exists a configuration z_i such that $y_i \asymp T_\sigma(z_i)$, namely, $y_i|_{G-B'} = T_\sigma(z_i)|_{G-B'}$ for some finite subset $B' \subset G$. Clearly, there exist infinitely many $h \in H$ for which we have $y_i|_{hA_i} =$

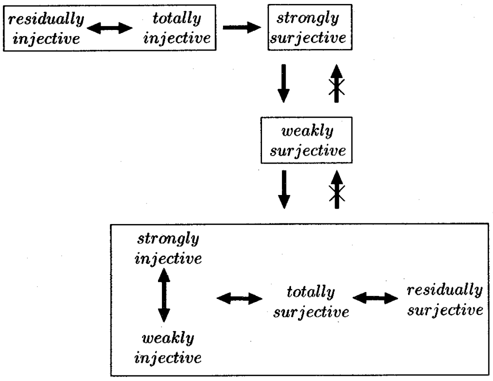

Fig. 7 The M-K hierarchy with GOE and residual finiteness.

$s_h(y|_{A_i}) = T_\sigma(z_i)|_{hA_i}$. By a shift, we have $y|_{A_i} = T_\sigma(s_{h^{-1}}z_i)|_{A_i}$, and consequently $y|_{K_i} = T_\sigma(s_{h^{-1}}z_i)|_{K_i}$. This means that $P_i \neq \emptyset$. □

9. Conclusions

Modified Maruoka and Kimura's hierarchies were given for the classes of non-Euclidean cellular automata of GOE type and residually finite type. If these two properties are both satisfied, we have a hierarchy that is approximately the same as Maruoka and Kimura's original hierarchy. We summarize it here as the fourth main theorem.

Theorem 4. Assume that G has the GOE property and is residually finite. For any local map σ, relations among properties of T_σ are summarized as in **Fig. 7**.

At present, we do not have an answer to the question whether total injectivity is equivalent to strong surjectivity.

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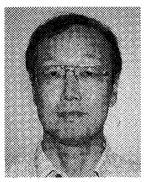

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