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S_5－FACTORIZATION ALGORITHM OF COMPLETE BIPARTITE GRAPHS

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1. Introduction

Let S_5 be a ***star*** on 5 vertices and $K_{m,n}$ be a ***complete bipartite graph*** with partite sets V_1 and V_2 of m and n vertices each. A spanning subgraph F of $K_{m,n}$ is called an S_5-***factor*** if each component of F is isomorphic to S_5. If $K_{m,n}$ is expressed as an edge-disjoint sum of S_5-factors, then this sum is called an S_5-***factorization*** of $K_{m,n}$.

2. S_5-factorization of $K_{m,n}$

Notation 1. **r,t,b** : number of S_5-factors, number of S_5-components of each S_5-factor, and total number of S_5-components, respectively, in an S_5-factorization of $K_{m,n}$.

t_1 (t_2) : number of components whose centers are in V_1 (V_2), respectively, among t S_5-components of each S_5-factor.

$r_1(u)$ ($r_2(v)$) : number of components whose centers are all u (v) for any u (v) in V_1 (V_2), respectively, among b S_5-components.

Lemma 1. If $K_{m,n}$ has an S_5-factorization then (i) $b=mn/4$, (ii) $t=(m+n)/5$, (iii) $r=5mn/4(m+n)$, (iv) $t_1=(4n-m)/15$, (v) $t_2=(4m-n)/15$, (vi) $r_1=(4n-m)n/12(m+n)$, (vii) $r_2=(4m-n)m/12(m+n)$, (viii) $m \leqq 4n$ and (ix) $n \leqq 4m$.

Lemma 2. If $K_{m,n}$ has an S_5-factorization, then $K_{sm,sn}$ has an S_5-factorization for every positive integer s.

3. S_5-factorization algorithm of $K_{m,n}$

Case (1) m=4n: From Lemma 2, $K_{4n,n}$ has an S_5-factorization since $K_{4,1}$ is just S_5.

Case (2) n=4m: Obviously, $K_{m,4m}$ has an S_5-factorization.

Case (3) m<4n and n<4m: Let $x=(4n-m)/15$ and $y=(4m-n)/15$. From Conditions (iv)-(v), x and y are integers such that $0<x<m$ and $0<y<n$. We have $x+4y=m$ and $4x+y=n$. It holds that $b=(x^2+4xy+y^2)+xy/4$, $t=x+y$, $r=(x+y)+9xy/4(x+y)$, $t_1=x$, $t_2=y$, $r_1=x-3xy/4(x+y)$ and $r_2=y-3xy/4(x+y)$. Let $z=3xy/4(x+y)$, which is a positive integer. And let $(x,4y)=d$, $x=dp$, $4y=dq$, where $(p,q)=1$. Then $dq/4$ is an integer and $z=3dpq/4(4p+q)$.

Lemma 3. Let $(p,q)=1$ and s be an integer. Then

(I) when q is an odd integer,

$m=4(p+q)(4p+q)s, n=(16p+q)(4p+q)s$ ($(4p+q)/3$:not integer)

or $m=4(p+q)(4p+q)s/3, n=(16p+q)(4p+q)s/3$ ($(4p+q)/3$:integer)

(II) when $q=2q'$ (q' is an odd integer),

$m=4(p+2q')(2p+q')s, n=2(8p+q')(2p+q')s$ ($(2p+q')/3$:not integer)

or $m=4(p+2q')(2p+q')s/3, n=2(8p+q')(2p+q')s/3$ ($(2p+q')/3$:integer)

(III) when $q=4q''$,

$m=4(p+4q'')(p+q'')s, n=4(4p+q'')(p+q'')s$ ($(p+q'')/3$:not integer)

or $m=4(p+4q'')(p+q'')s/3, n=4(4p+q'')(p+q'')s/3$ ($(p+q'')/3$:integer).

Notation 2. Let A and B be two sequences of the same size such as

A: $a_1, a_2, \ldots, a_u$

B: $b_1, b_2, \ldots, b_u$.

If $b_i=a_i+c$ $(i=1,2,...,u)$, then we write $B=A+c$. If $b_i=((a_i+c) \bmod w)$ $(i=1,2,...,u)$, then we write $B=A+c \bmod w$, where the residuals $a_i+c \bmod w$ are integers in the set $\{1,2,...,w\}$.

Lemma 4. $(p,q)=1$ and q is an odd integer
$m=4(p+q)(4p+q)s$, $n=(16p+q)(4p+q)s$
$\Longrightarrow$ $K_{m,n}$ has an S_5-factorization.

Proof.(Algorithm I) Put $s=1$. Let $x=(4n-m)/15$, $y=(4m-n)/15$, $t=(m+n)/5$, $r=5mn/4(m+n)$. Then $x=4p(4p+q)$, $y=q(4p+q)$, $t=(4p+q)^2$, $r=(p+q)(16p+q)$. Let $r_m=p+q$, $r_n=16p+q$, $m_0=m/r_m=4(4p+q)$, $n_0=n/r_n=4p+q$.
Consider two sequences R and C of the same size $16(4p+q)$.
R: $1,1,1,1,2,2,2,2,...,4(4p+q),4(4p+q),4(4p+q),4(4p+q)$
C: $1,2,...,16(4p+q)-1,16(4p+q)$.
Construct $R_i=R+4(i-1)(4p+q)$ $(i=1,2,...,p)$.
Construct $C_i=(C+4(i-1) \bmod 16(4p+q))+16(i-1)(4p+q)$ $(i=1,2,...,p)$.
Consider two sequences R' and C' of the same size $4(4p+q)$.
R': $r_1,r_2,...,r_{4(4p+q)}$, where $r_i=(i-1)p+1 \bmod 4(4p+q)$ $(i=1,2,...,4(4p+q))$
C': $c_1,c_2,...,c_{4(4p+q)}$, where $c_i=n-(i-1)q \bmod q(4p+q)$ $(i=1,2,...,4(4p+q))$.
Construct $R_i'=R'+4(i-1)(4p+q)+4p(4p+q)$ $(i=1,2,...,q)$.
Construct $C_i'=(C'-(i-1) \bmod q(4p+q))+16p(4p+q)$ $(i=1,2,...,q)$.
Consider two sequences I and J of the same size $4t$.
I: $R_1,R_2,...,R_p,R_1',R_2',...,R_q'$
J: $C_1,C_2,...,C_p,C_1',C_2',...,C_q'$.
Let i_k and j_k be the k-th element of I and J, respectively $(k=1,2,...,4t)$. Join two vertices i_k in V_1 and j_k in V_2 with an edge (i_k,j_k) $(k=1,2,...,4t)$. Construct a graph F with two vertex sets $\{i_k\}$ and $\{j_k\}$ and an edge set $\{(i_k,j_k)\}$. Then F is an S_5-factor of $K_{m,n}$. This graph is called an ***S_5-factor constructed with two sequences I and J.***
Construct I_i such that $I_i=I+(i-1)m_0 \bmod m$ $(i=1,2,...,r_m)$.
Construct J_j such that $J_j=J+(j-1)n_0 \bmod n$ $(j=1,2,...,r_n)$.
Construct S_5-factors F_{ij} with I_i and J_j $(i=1,2,...,r_m; j=1,2,...,r_n)$. Then F_{ij} are edge-disjoint and their sum is an S_5-factorization of $K_{m,n}$. By Lemma 2, $K_{m,n}$ has an S_5-factorization for every positive integer s. □

Lemma 5. $(p,q)=1$ and $q=2q'$ (q' is an odd integer)
$m=4(p+2q')(2p+q')s$, $n=2(8p+q')(2p+q')s$
$\Longrightarrow$ $K_{m,n}$ has an S_5-factorization.

Lemma 6. $(p,q)=1$ and $q=4q''$
$m=4(p+4q'')(p+q'')s$, $n=4(4p+q'')(p+q'')s$
$\Longrightarrow$ $K_{m,n}$ has an S_5-factorization.

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