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Efficient Algorithms for Edge-Coloring Partial k-Trees

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Abstract

Many combinatorial problems can be efficiently solved for partial k-trees. The edge-coloring problem is one of a few combinatorial problems for which no efficient algorithms have been obtained for partial k-trees. This report presents two algoritms. One decides the chromatic index of a given partial k-tree in linear time. The other optimally edge-colors a given partial k-tree G in $O(|V|^2)$ time, where V is the set of vertices of G. In the report k is assumed to be a constant.

1. Introduction

This report deals with the edge-coloring problem which asks to color, using a minimum number of colors, all edges of a given graph so that no two adjacent edges are colored with the same color. The chromatic index $\chi'(G)$ of a graph G is the minimum number of colors used by an edge-coloring of G. This problem arises in many applications including various scheduling and partitioning problems [FW]. Since the edge-coloring problem is NP-complete [Hol], it seems unlikely that there exists a polynomial-time algorithm for the problem on general graphs. On the other hand, it is known that many combinatorial problems can be solved very efficiently, say in linear time, for partial k-trees [TNS, ACPD, AL, C]. Such a class of problems has been characterized in terms of "forbidden graphs" or "extended monadic logic of second order" [TNS, ACPD, AL, C]. The edge-coloring problem does not belong to such a class, and is one of a few well-known problems for which no efficient algorithms of low complexity order have been obtained for partial k-trees. Even a polynomial-time algorithm has not been obtained for the edge-coloring problem of partial k-trees until recently Bodlaender give a polynomial-time algorithm of high complexity order [Bod]. The complexity of his algorithm is $O(|V|\Delta^{2^{2(k+1)}})$, where Δ is the maximum degree of G. Note that the maximum degree Δ is not a constant in general.

In the report we give two algorithms. One determines the chromatic index $\chi'(G)$ of a given partial k-tree G in linear time. The other finds an edge-coloring of G using $\chi'(G)$ colors in $O(|V|^2)$ time.

2. Terminology and definitions

In this section we give some definitions. Let $G=(V,E)$ be a given graph with vertex set V and edge set E. In this report we consider only simple graphs, which have no multiple or self-loop edges. A partial k-tree is defined as follows.

Definition 2.1 *The class of k-trees is defined recursively as follows.*

1. *A complete graph with k vertices is a k-tree.*
2. *If $G=(V,E)$ is a k-tree, $v_1, v_2, ..., v_k$ induce a complete subgraph of G with k vertices, and $w \notin V$, then $H=(V\cup\{w\}, E\cup\{(v_i,w) | 1\le i\le k\})$ is a k-tree.*
3. *All k-trees can be formed with rules 1 and 2.*

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Definition 2.2 *A graph is a partial k-tree if it is a subgraph of a k-tree.*

We denote by $d(v)$ the degree of vertex $v \in V$. The maximum degree of G is denoted by $\Delta(G)$ or simply by Δ. The number of vertices which have degree Δ and are adjacent with vertex v is denoted by $n_\Delta(v)$. An edge joining vertices u and v is denoted by (u,v). The graph obtained from G by deleting an edge (u,v) is denoted by $G-(u,v)$. Similarly define $G+(u,v)$. An edge (u,v) of G is defined to be *eliminable* if

$$d(u)+n_\Delta(v)\le\Delta \quad \text{when } d(u)<\Delta\text{; and}$$
$$n_\Delta(v)=1 \quad \text{when } d(u)=\Delta.$$

3. Edge-coloring algorithms

Hoover [Hoo] has claimed that the chromatic index of partial k-trees can be determined in linear time. However his proof is incorrect. Indeed his result is based on "Theorem 4.5" in [Hoo]: if the chromatic index of a general graph G is $\Delta(G)+1$ then

$$|E|\ge\frac{|V|\cdot\Delta(G)}{4}.$$

In fact this "Theorem" is incorrect; there is a counterexample as follows. Let G be a graph obtained from K_7, a complete graph of seven vertices, by inserting seventy vertices on an arbitrary edge e of K_7. Then $\Delta(G)=6$, $|V|=77$ and $|E|=91$. Clearly $\chi'(G)=\Delta(G)+1=7$ since $\chi'(K_7-e)=7$. Therefore

$$|E|<\frac{|V|\cdot\Delta(G)}{4}$$

contrary to the "Theorem." This flaw comes from a misinterpretation of a result on "critical graphs" in [FW].

We fix the flaw and give correct algorithms. We first give a lemma.

Lemma 3.1 *A partial k-tree G has an eliminable edge if $\Delta\ge 2k$.*

Proof. Since G is a partial k-tree, G has a vertex of degree at most k. Let S be the set of such vertices, and let $G'=G-S$ be the graph obtained from G by deleting all vertices in S. Since G' is also a partial k-tree, G' has a vertex v of degree at most k. Since the degree of v was at least $k+1$ in G, v was adjacent with a vertex $u\in S$ in G. Therefore we have $n_\Delta(v)\le k$, and hence $d(u)+n_\Delta(v)\le 2k$. Thus the edge (u,v) is eliminable. $\mathcal{Q.E.D.}$

The following lemma has been known [TN,NC].

Lemma 3.2 *If (u,v) is an eliminable edge of a simple graph G, then*

$$\chi'(G) = \max\{\Delta(G), \chi'(G-(u,v))\}. \qquad \Box$$

We then have the following theorem.

Theorem 3.3 *If G is a partial k-tree and $\Delta(G) \geq 2k$, then $\chi'(G) = \Delta(G)$.*
Proof. By Lemma 3.1 G has an eliminable edge. Delete from G eliminable edges $e_1, e_2, \cdots, e_j$ sequentially until $G' = G - \{e_1, e_2, \cdots, e_j\}$ satisfies $\Delta(G') = \Delta(G) - 1$. Then $\chi'(G') \leq \Delta(G') + 1 = \Delta(G)$ by Vizing's theorem [FW, NC]. By Lemma 3.2 $\chi'(G) = \max\{\Delta(G), \chi'(G')\}$. Therefore $\chi'(G) = \Delta(G)$. $\mathcal{Q.E.D.}$

Lemma 3.4 *The edge-coloring problem can be solved in $O(|V|)$ time using $O(|V|)$ space for a partial k-tree G if $\Delta(G) < 2k$.*
Proof. Bodlaender [Bod] has given an algorithm to solve the edge-coloring problem for a partial k-tree G in $O(|V| \cdot \Delta^{2^{2(k+1)}})$ time using space of the same order. Since $\Delta(G) < 2k$, the complexity is $O(|V|)$. $\mathcal{Q.E.D.}$

We now have the following theorem.

Theorem 3.5 *The chromatic index of partial k-trees can be determined in linear time.*
Proof. We can compute the maximum degree $\Delta(G)$ of a given partial k-tree G in linear time. If $\Delta(G) \geq 2k$, then $\chi'(G) = \Delta(G)$ by Theorem 3.3; otherwise, $\chi'(G)$ can be determined in linear time by Lemma 3.4. $\mathcal{Q.E.D.}$

The following Lemma 3.6 is known [TN, NC].

Lemma 3.6 *Let G be a graph and (u,v) be an eliminable edge in G. If an edge-coloring of $G-(u,v)$ with $\Delta(G)$ colors is given, then an edge-coloring of G with $\Delta(G)$ colors can be obtained in $O(|V|)$ time using $O(|E|)$ space.* $\Box$

We now give an algorithm which edge-colors a partial k-tree G with $\chi'(G)$ colors.

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Procedure Color(G);
begin
1  if Δ(G) < 2k then
2    find an edge-coloring of G using χ'(G) colors by
       Bodlaender's algorithm                 { Lemma 3.4 }
3  else { Δ(G) ≥ 2k }
4  begin
5    G':=G;
6    while Δ(G') = Δ(G) do
7    begin
8      find an eliminable edge (u,v) in G';
9      G':=G' − (u,v);                        {delete (u,v)}
10     push (u,v) on the top of stack S
11   end;
12   {Δ(G') = Δ(G) − 1}
13   color G' with Δ(G) = Δ(G') + 1 colors;
14   while stack S is not empty do
15   begin
16     pop up an edge, say (u,v), from S;
17     G' := G' + (u,v);
18     update the coloring of G' from that of G' − (u,v)
                                              { Lemma 3.6 }
19   end
20 end
end;
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Line 2 in the algorithm above can be done in $O(|V|)$ time by Lemma 3.4. Finding eliminable edges in Line 8 can be done total in $O(|V||E|)$ time [TN, NC]. Since $|E| \leq k|V|$, it can be done in $O(|V|^2)$ time. Terada and Nishizeki [TN] gave an algorithm to edge-color a general graph $G = (V, E)$ with Δ or $\Delta + 1$ colors in $O(|E||V|)$ time. Gabow et al. improved the time complexity to be $O(|E|\sqrt{|V|\log|V|})$ [GNKLT]. Therefore Lines 13 can be done in $O(|V|\sqrt{|V|\log|V|})$ time. By Lemma 3.6 Line 14-19 can be done in $O(|V|^2)$ time. Thus the total running time of the algorithm is $O(|V|^2)$. Hence we conclude:

Theorem 4.7 *The edge-coloring problem can be solved in $O(|V|^2)$ time for a partial k-tree G if k is a constant.*

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