CYCLIC S_K-FACTORIZATION ALGORITHMS OF COMPLETE BIPARTITE GRAPHS

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Abstract. In this paper, trivial necessary conditions and base conditions for the existence of an S_K-factorization of $K_{m,n}$ are given. Cyclic S_K-factorization algorithms of $K_{m,n}$ are also given.

1. Introduction. Let S_K be a *star* on k vertices and $K_{m,n}$ be a *complete bipartite graph* with partite sets V_1 and V_2 of m and n vertices each. A spanning subgraph F of $K_{m,n}$ is called an *S_K-factor* if each component of F is isomorphic to S_K. If $K_{m,n}$ is expressed as an edge-disjoint sum of S_K-factors, then this sum is called an *S_K-factorization* of $K_{m,n}$. Moreover, If we can choose a special S_K-factor F such that an S_K-factorization of $K_{m,n}$ is obtained by cyclic shifting of vertices of F, then this special factor is called a *base factor* and this factorization is called a *cyclic S_K-factorization*.

2. Cyclic S_K-factorization of $K_{m,n}$.

Notations. r,t,b : number of S_K-factors, number of S_K-components of each S_K-factor, and total number of S_K-components, respectively, in an S_K-factorization of $K_{m,n}$.

t_1 (t_2) : number of components whose centers are in V_1 (V_2), respectively, among t S_K-components of each S_K-factor.

$r_1(u)$ ($r_2(v)$) : number of components whose centers are all u (v) for any u (v) in V_1 (V_2), respectively, among b S_K-components.

Trivial necessary conditions. $b=mn/(k-1)$, $t=(m+n)/k$, $r=kmn/(k-1)(m+n)$, $t_1=\{(k-1)n-m\}/k(k-2)$, $t_2=\{(k-1)m-n\}/k(k-2)$, $r_1=\{(k-1)n-m\}n/(k-1)(k-2)(m+n)$, $r_2=\{(k-1)m-n\}m/(k-1)(k-2)(m+n)$, $m \leqq (k-1)n$, $n \leqq (k-1)m$.

Base conditions. $m=r_m m_0$, $n=r_n n_0$, $r=r_m r_n$, $t_1=pm_0$, $t_2=qn_0$.

Rectangle area. Use a rectangle area of size m by n whose (i,j) entry denotes an edge joining u_i in V_1 and v_j in V_2. Then an S_K-factor has t_1 H-type S_K's and t_2 V-type S_K's, where H-type S_K (V-type S_K) is an S_K whose center is in V_1 (V_2), respectively. Divide this rectangle area into four rectangle subareas A,B,C,D whose sizes are t_1 by $(k-1)t_1$, t_1 by t_2, $(k-1)t_2$ by $(k-1)t_1$, and $(k-1)t_2$ by t_2, respectively.

Base factor. Choose a special S_K-factor F whose t_1 H-type S_K's are in A and t_2 V-type S_K's are in D. Shift F right cyclically step n_0. Then we have r_n S_K-factors. Shift those r_n S_K-factors down cyclically step m_0. Then we have $r_m r_n(=r)$ S_K-factors. If those r S_K-factors cover A,B,C,D neither too much nor too less, then this special S_K-factor is a base factor. And the sum of those r S_K-factors is a cyclic S_K-factorization of $K_{m,n}$.

Lemma 1. Trivial necessary conditions, base conditions, and $(r_n-q)/p$ is an integer
===> $K_{m,n}$ has a cyclic S_K-factorization.

Proof. About vertical size and horizontal size of A it holds that $t_1=pm_0$ and $(k-1)t_1=p(k-1)m_0$ $=p\{(r_n-q)/p\}n_0$. Since $(r_n-q)/p$ is an integer, divide A into p^2 rectangle subareas A_{ij} of size m_0 by $(k-1)m_0$ each. In A_{11}, take m_0 H-type S_K's as following: diagonally in A_{11}, (k-1)-right diagonally in A_{22}, 2(k-1)-right diagonally in A_{33}, and so on. Then we have $pm_0(=t_1)$ H-type S_K's in A.

About vertical size and horizontal size of D it holds that $(k-1)t_2=(k-1)qn_0=(r_m-p)m_0$ and $t_2=qn_0$. Divide D into (r_m-p) rectangle subareas D_i of size m_0 by qn_0 each.

We consider three subcases as follows: (a.1) m_0/p and $\{n_0-(k-1)p\}m_0/pqn_0$ are integers, (a.2) m_0/p is an integer and $\{n_0-(k-1)p\}m_0/pqn_0$ is not an integer, (a.3) m_0/p is not an integer.

Case (a.1). m_0/p and $\{n_0-(k-1)p\}m_0/pqn_0$ are integers. In D_1, use stepwise continuous boxes, each box is a area of size 1 by $n_0-(k-1)p$, and vertical (k-1)-lines with $n_0-(k-1)p$ wide. Take m_0 boxes which are horizontally continuous and vertically step-continuous with step p or p+1.

Then the entries on crossing points of the stepwise continuous boxes and each vertical (k-1)-line form a V-type S_K, i.e. , one V-type S_K appears on each vertical (k-1)-line. In D_1, D_2, D_3,..., shift vertical (k-1)-lines right simultaneously one by one. Then we have $qn_0(=t_2)$ V-type S_K's in D.

Case (a.2). m_0/p is an integer and $\{n_0-(k-1)p\}m_0/pqn_0$ is not an integer. In D_1, take stepwise continueous boxes and vertical (k-1)-lines. Then similarly as in Case (a.1), we have $qn_0(=t_2)$ V-type S_K's in D.

Case (a.3). m_0/p is not an integer. Let $(m_0,p)=d$. Put $m_0=dm_1$, $p=dp_1$, where $(m_1,p_1)=1$. In D_1, take dm_1 horizontally continuous boxes such as first m_1 boxes started at the first row, second m_1 boxes started at the second row, ... , and last m_1 boxes started at the d-th row are vertically step-continuous with step p. Then similarly as in Case (a.1) and (a.2), we have $qn_0(=t_2)$ V-type S_K's in D.

It can be easily checked that t_1 H-type S_K's in A and t_2 V-type S_K's in D form a base factor. Therefore, $K_{m,n}$ has a cyclic S_K-factorization. ■

Lemma 2. Trivial necessary conditions, base conditions, and $(r_m-p)/q$ is an integer

===> $K_{m,n}$ has a cyclic S_K-factorization.

Lemma 3. Trivial necessary conditions, base conditions, and $(r_n-q)/p$ and $(r_m-p)/q$ are not integers

===> $K_{m,n}$ has an S_K-factorization.

Theorem. Trivial necessary conditions and base conditions

===> $K_{m,n}$ has an S_K-factorization.

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