あいまいな識別不可能性の代数構造について

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1. Introduction

Pawlak [8] proposed a mathematical framework, i.e., the so called information system, to formalize the knowledge representation. Later, several rearchers introduced fuzzy concepts into the information system based on the work of Pawlak, e.g., fuzzy rough sets [2, 3, 5], rough fuzzy sets [2] and a fuzzy modal logic [6]. In addition, the notion, fuzziness in indiscernibility, was suggested in [3].

This paper will provide a definition of fuzzy indiscernibility in a variant of the information system and an algebraic approach to it.

2. Basic notions

2.1. The following operations on [0,1] follow from [4]: for $p, q\in [0,1]$, $p\wedge q=\max(0,p+q-1)$, $p\vee q=\min(1,p+q)$, $p\rightarrow q=\min(1,1-p+q)$.

2.2. A *fuzzy information system* is the quadruple $I=(OB,FA,[0,1],g)$ where OB is an set of objects; FA is a set of fuzzy attributes of the objects, e.g., tall; $g:OB\times FA\rightarrow[0,1]$.

2.3. For $A\subseteq FA$, a *fuzzy indiscernibility relation* $\tilde{A}(X)$ is a fuzzy subset of $X\times X\subseteq OB\times OB$ characterized by membership function μ (cf.[9]):

$$\mu_{\tilde{A}(X)}(o_1,o_2)= \bigwedge_{a\in A} (g(o_1,a)\leftrightarrow g(o_2,a)) \text{ for } o_1, o_2 \in X,$$

$\tilde{\emptyset}(X)=X\times X$, i.e., $\mu_{\tilde{\emptyset}(X)}(o_1,o_2)=1$ for $o_1, o_2\in X$.

Let $FI(X,A)=\{\tilde{B}(X): B\subseteq A\}$.

2.4. For any $A\subseteq FA$, we can show the following.

1) $\mu_{\tilde{A}(X)}(o,o)=1$.

2) $\mu_{\tilde{A}(X)}(o_1,o_2)=\mu_{\tilde{A}(X)}(o_2,o_1)$.

3) $\max_{o\in X}\{\mu_{\tilde{A}(X)}(o_1,o)\wedge\mu_{\tilde{A}(X)}(o,o_2)\}\leq\mu_{\tilde{A}(X)}(o_1,o_2)$.

2.5. 1) For $B\subseteq A\subseteq FA$ $\neg\tilde{B}(X)=\widetilde{A\text{-}B}(X)$, called (X,A)-*complement* of $\tilde{B}(X)$.

2) $\tilde{A}(X)\cap\tilde{B}(X)$ is defined by

$$\mu_{\tilde{A}(X)\cap\tilde{B}(X)}(o_1,o_2)= \bigwedge_{a\in A\cup B} \mu_{\widetilde{\{a\}}}(o_1,o_2).$$

3) $\tilde{A}(X)\Rightarrow\tilde{B}(X)$ is characterized by

$$\mu_{\tilde{A}(X)\Rightarrow\tilde{B}(X)}=\max\{\mu_{\tilde{C}(X)}(o_1,o_2)\in[0,1]:\ \mu_{\tilde{A}(X)\cap\tilde{C}(X)}(o_1,o_2)\leq\mu_{\tilde{B}(X)}(o_1,o_2)\}.$$

Note that $\cap$ is a symmetric operation.

2.6. The relations $\subseteq$ and $=$ on FI(X,A) are defined as follows:

1) $\tilde{B}(X)\subseteq\tilde{C}(X)$ iff $\mu_{\tilde{B}(X)}(o_1,o_2)\leq\mu_{\tilde{C}(X)}(o_1,o_2)$ for all $o_1, o_2\in X$;

2) $\tilde{B}(X)=\tilde{C}(X)$ iff $\mu_{\tilde{B}(X)}(o_1,o_2)=\mu_{\tilde{C}(X)}(o_1,o_2)$ for all $o_1, o_2\in X$.

2.7. 1) $=$ is not a congruence on FI(X,A).

2) $\subseteq$ is not a congruence on FI(X,A).

3) If $B\subseteq C$, then $\tilde{C}(X)\subseteq\tilde{B}(X)$; but the converse does not hold.

Further, By using the fuzzy indiscernibility, we classify the fuzzy attributes in the following ways.

2.8. 1) A set A of fuzzy attributes is X-*dependent* in an fuzzy information system I if there is a set C of fuzzy attributes in S such that $C\subseteq A$ and $\widetilde{A\text{-}C}(X)=\tilde{A}(X)$.

2) A is X-*independent* in I if A is not X-dependent in I.

3) A is X-*superfluous* in the set B iff $\widetilde{B\text{-}A}(X)=\tilde{B}(X)$.

3. FIS-algebra for fuzzy information system

In the following, we will assume that I=(OB, FA, [0,1],g) is a fixed fuzzy information system. For $X\subseteq OB$ and $A\subseteq FA$ FI(X,A) is closed under $\neg$, $\cap$ and $\Rightarrow$. Denote $X\times X$ by T_X.

3.1. By an FIS-*algebra* we mean the structure $(FI(X,A), \neg, \cap, \Rightarrow, T_X)$, denoted by **FI**(X,A), where T_X is called the unit element with respect to X.

3.2. The following statements are the axioms for the class of FIS-algebras **FI**(X,A).

Ax1. $\tilde{B}(X)\Rightarrow\tilde{B}(X)=T_X$.

Ax2. $\neg\tilde{B}(X)=\widetilde{A\text{-}B}(X)$.

Ax3. $\tilde{B}(X)\cap T_X=\tilde{B}(X)$.

Ax4. $\widetilde{B\cup C}(X)=\tilde{B}(X)\cap\tilde{C}(X)$.

Ax5. $\tilde{B}(X)\cap\tilde{D}(X)\subseteq\tilde{C}(X)$ iff $\tilde{D}(X)\subseteq\tilde{B}(X)\Rightarrow\tilde{C}(X)$.

Ax6. $(\tilde{B}(X)\cap\neg\tilde{B}(X))\Rightarrow\tilde{C}(X)=T_X$.

Ax7. $\tilde{B}(X)\cap\neg\tilde{B}(X)=\neg(\tilde{B}(X)\Rightarrow\tilde{B}(X))$.

3.3. The following elementary properties of an FIS-algebra **FI**(X,A) hold:

1) $\neg(\tilde{B}(X)\Rightarrow\tilde{B}(X))\Rightarrow\tilde{C}(X)=T_X$;

2) $\tilde{B}(X)\Rightarrow\tilde{C}(X)=T_X$ iff $\tilde{B}(X)\subseteq\tilde{C}(X)$;

3) $\tilde{C}(X)\subseteq\tilde{B}(X)\Rightarrow\tilde{C}(X)$ iff $\tilde{C}(X)\subseteq\tilde{B}(X)$ and $\tilde{B}(X)\neq\tilde{C}(X)$;

4) $(\tilde{B}(X)\Rightarrow\tilde{C}(X))\cap\tilde{C}(X)\subseteq\tilde{C}(X)$ and

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$\tilde{B}(X)\cap\tilde{C}(X)\subseteq\tilde{B}(X)\cap(\tilde{B}(X)\Rightarrow\tilde{C}(X))$;

5) $\tilde{B}(X)\cap\tilde{C}(X)\subseteq\tilde{B}(X)$ and $\tilde{B}(X)\cap\tilde{C}(X)\subseteq\tilde{C}(X)\}$.

3.4. Given an FIS-algebra **FI(X,A)**, define that Um(FI(X,A))=$\cup$\{FI(X,B): B$\subseteq$A and B is not X-superfluous in A\} and if A=$\emptyset$, then Um(FI(X,A))=$\emptyset$. We call Um(FI(X,A)) universe of the subalgebra of **FI(X,A)** with maximal X-independent set of fuzzy attributes.

3.5. 1) Um(Um(FI(X,A)))=Um(FI(X,A)).

2) A$\subseteq$B implies Um(FI(X,A))$\subseteq$Um(FI(X,B)).

3) Um(FI(X,A))$\cup$Um(FI(X,B))=Um(FI(X,A$\cup$B)).

4) Um(FI(X,A$\cap$B))$\subseteq$Um(FI(X,A))$\cap$Um(FI(X,B)).

3.6. Let $\lambda=\frac{\text{Card(Um(FI(X,A)))}}{\text{Card(FI(X,A))}}$. If A is X-independent, then $\lambda=1$.

4. Representation theorem for FIS-algebra

We will show the correspondence of an FIS-algebra to a normal algebraic structure, i.e., the representation of a FIS-algebra (cf.[7]).

4.1. For $\alpha\in[0,1]$, $\tilde{A}(X)_\alpha=\{(o,o')\in X\times X: \mu_{\tilde{A}(X)}(o,o')\geq\alpha\}$. By Zadeh [9], $\tilde{A}(X)_\alpha$ is called an α-*level-set* of $\tilde{A}(X)$. FL(X,A)=$\{\{\tilde{B}(X)_\alpha\}_{\alpha\in[0,1]}: B\subseteq A\}$.

4.2. 1) The relation $\propto$ on FL(X,A) is defined by

$\{\tilde{B}(X)_\alpha\}_{\alpha\in[0,1]}\propto\{\tilde{C}(X)_\alpha\}_{\alpha\in[0,1]}$ iff for $\alpha\in[0,1]$ $\tilde{B}(X)_\alpha\subseteq\tilde{C}(X)_\alpha$.

2) The operation $\otimes$ on FL(X,A) is defined by

$\{\tilde{B}(X)_\alpha\}_{\alpha\in[0,1]}\otimes\{\tilde{C}(X)_\alpha\}_{\alpha\in[0,1]}=\{(\tilde{B}(X)\cap\tilde{C}(X))_\alpha\}_{\alpha\in[0,1]}$.

3) The operation © on FL(X,A) is defined by

$\{\tilde{B}(X)_\alpha\}^{©}_{\alpha\in[0,1]}=\{(\neg\tilde{B}(X))_\alpha\}_{\alpha\in[0,1]}$.

4) The operation $\ominus$ on FL(X,A) is defined by

$\{\tilde{B}(X)_\alpha\}_{\alpha\in[0,1]}\ominus\{\tilde{C}(X)_\alpha\}_{\alpha\in[0,1]}$
$=\sup\{\{\tilde{D}(X)_\alpha\}_{\alpha\in[0,1]}:$
$\{\tilde{B}(X)_\alpha\}_{\alpha\in[0,1]}\otimes\{\tilde{D}(X)_\alpha\}_{\alpha\in[0,1]}$
$\propto\{\tilde{C}(X)_\alpha\}_{\alpha\in[0,1]}\}$.

Let $I_X=\{X\times X\}$. Then, $I_X\in$FL(X,A) since $\tilde{I_X}=\{\emptyset_\alpha\}_{\alpha\in[0,1]}$.

4.3.(Representation Theorem) The algebra (FI(X,A), $\neg$, $\cap$, $\Rightarrow$, T_X) and the algebra (FL(X,A), ©, $\otimes$, $\ominus$,I_X\} are isomorphic.

5. Terms

5.1. Let V be a set of variables. The set T(V) of *terms* for FIS-algebras is the least set satisfying the following conditions:

1) V$\subseteq$T(V);

2) if $t_1,t_2\in$ T(V), then $\neg t_1, t_1\cap t_2, t_1\Rightarrow t_2\in$ T(V).

For a term t, we will write $t(x_1,x_2,...,x_n)$ where x_1, x_2, ..., x_n are all the variables occurring in the term t.

5.2. Given a term $t(x_1,x_2,...,x_n)$ and an FIS-algebra **FI(X,A)**, we define a valuation function $t^{\mathbf{FI(X,A)}}$ assigning t into FI(X,A):

1) for t$\in$ V, $t^{\mathbf{FI(X,A)}}\in$ FI(X,A);

2) $(\neg t)^{\mathbf{FI(X,A)}}=\neg(t^{\mathbf{FI(X,A)}})$;

3) $(t_1\cap t_2)^{\mathbf{FI(X,A)}}=t_1^{\mathbf{FI(X,A)}}\cap t_2^{\mathbf{FI(X,A)}}$;

4) $(t_1\Rightarrow t_2)^{\mathbf{FI(X,A)}}=t_1^{\mathbf{FI(X,A)}}\Rightarrow t_2^{\mathbf{FI(X,A)}}$.

For brevity, we write $t(x_1^{\mathbf{FI(X,A)}},x_2^{\mathbf{FI(X,A)}},...,x_n^{\mathbf{FI(X,A)}})$ for $(t(x_1,x_2,...,x_n))^{\mathbf{FI(X,A)}}$.

6. The center of an FIS-algebra

6.1. Let **FI(X,A)** be an FIS-algebra. Then, the *center* (cf. [1]) of **FI(X,A)** is the binary relation **Z(FI(X,A))** such that

Z(FI(X,A))=$\{(\tilde{B}(X),\tilde{B}'(X))\in$ FI(X,A)$\times$FI(X,A): for every t$\in$ T(V) $t(\tilde{B}(X),\Gamma)=t(\tilde{B}(X),\Delta)$ iff $t(\tilde{B}'(X),\Gamma)=t(\tilde{B}'(X),\Delta)\}$

where Γ, $\Delta\in FI(X,A)^{n-1}$ if t is of the form $t(x_1,x_2,...,x_n)$.

6.2. 1) **Z(FI(X,A))** is an equivalence relation on FI(X,A).

2)**Z(FI(X,A))** is a congruence on FI(X,A).

6.3. **Z(FI(X,A))**=$\{(\tilde{B}(X),\tilde{B}'(X))\in$ FI(X,A)$\times$FI(X,A): 1) B=B'; or 2) B$\cap$B'=B and B'-B is superfluous in B'; or 3) B$\cap$B'=B' and B-B' is superfluous in B; or 4) B and B' are superfluous in B$\cup$B'\}

7. Future works

As future works,we will consider the following topics:

1) Algebraic properties of the fuzzy indiscernibility in the case where **2.5**(2) is not necessary;

2) Correspondence of this algebraic study to fuzzy logical systems.

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