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Relationship Between Logic And Type System

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1. Introduction

The relationship between logics and type systems à la Church is investigated along the lines of H. Geuvers and H. Barendregt [1]. The used interpretation is based on *formulae-as-types* notion [2] but differs from [1]. The difference has made it possible to arrive at inverse interpretations for a restricted class of types. The scope of the interpretation is wider than the existing one in the sense that it may also be applied to an assumption with a proof figure to get a context constructively. The soundness and completeness results are given for eight logic and type systems. For this reason whole logical systems can be translated into type systems.

2. Logic and type system

We breifly present logic L and type system λL whose complete descriptions are in [3].

2-1. Higher order intuitionistic many sorted predicate logic

Term t and *sort S* are defined as follows. A set of terms t, sorts S , sorts St, and sorts Sp are denoted by **TERM**, **SORT**, **SORT-t** and **SORT-p** respectively:

$t ::= x \mid c \mid \lambda x.t \mid tt \mid t \supset t \mid \forall x \in S.t$

$S ::= St \mid Sp$

$St ::= A \mid St \to St \mid Sp \to St$

$Sp ::= \Omega \mid St \to Sp \mid Sp \to Sp$

A sort $[t] \in$ **SORT** is attached to each term $t \in$ **TERM**. A *formula* ϕ is defined by a term whose sort $[\phi]$ is Ω where [ ]:**TERM**→**SORT**. *Inference rules* are defined as follows and a *proof figure P* is constructed according to one of the inference rules:

$$\text{(assumption)}: \frac{\phi \in \Gamma}{\Gamma \vdash \phi}$$

$$(\supset I): \frac{\{\phi\} \cup \Gamma \vdash \psi}{\Gamma \vdash \phi \supset \psi}$$

$$(\supset E): \frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi \supset \psi}{\Gamma \vdash \psi}$$

$$(\forall I): \frac{\Gamma \vdash \psi}{\Gamma \vdash \forall x \in S.\psi}$$

※ $x \notin V(\Gamma)$, $[x] = S$

$$(\forall E): \frac{\Gamma \vdash \forall x \in S.\psi}{\Gamma \vdash \psi[x := t]}$$

※ $[t] = S$

2-2. λPK2 for $*_p$

Term M, type A, kind K, and *context* Γ are defined as follws:

$M ::= z \mid \lambda x{:}A_l.M \mid MM \mid \lambda X_p{:}K_p.M \mid MA_p$

$z ::= x \mid y \qquad l ::= p \mid t$

$A ::= A_p \mid A_t \mid A^{(i)} \mid A^{[j]}$

$A_p ::= Z_p \mid \Pi x{:}A_p.A_p \mid \Pi x{:}A_t.A_p \mid A^{(i)}M^i$
$\qquad \mid \Pi X_p{:}K_p.A_p \mid A^{[j]}A^j$

$A^{(1)} ::= \lambda x{:}A_t.A_p \qquad A^{(i+1)} ::= \lambda x{:}A_t.A^{(i)}$

$M^i ::= MM \cdots M$ i-fold $\qquad i ::= 1 \mid 1+i$

$A^{[1]} ::= \lambda X_p{:}K_p.A_p \qquad A^{[j+1]} ::= \lambda X_p{:}K_p.A^{[j]}$

$A^j ::= AA \cdots A$ j-fold $\qquad j ::= 1 \mid 1+j$

$A_t ::= X_t \mid \Pi x{:}A_t.A_t \mid \Pi X_p{:}K_p.A_t$

$Z ::= Z_p \mid Z_t \qquad Z_p ::= X_p \mid Y_p \qquad Z_t ::= X_t$

$K ::= K_p \mid K_t$

$K_p ::= *_p \mid \Pi x{:}A_t.K_p \mid \Pi X_p{:}K_p.A_p$

$K_t ::= *_t$

$* ::= *_p \mid *_t$

$\Gamma ::= < \ > \mid \Gamma, z{:}A \mid \Gamma, Z{:}K$

A complete presentation of the formulation rules for the above objects is omitted due to lack of space.

3. Interpretation

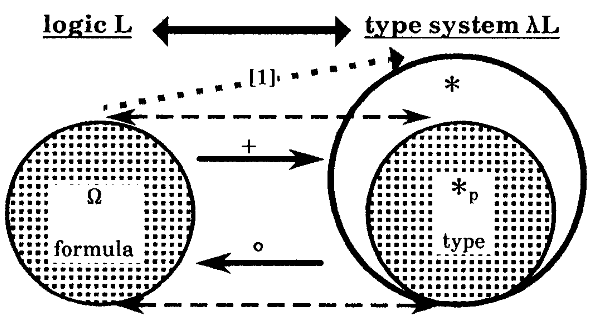


figure-1

Inerpretations + from L to λL and ° from λL to L are based on formulae-as-types notion. We connect predicate symbols to certain types, and sort Ω to

kind $*_p$ which is the subset of $*$. This point makes differences between $+$ and [1], and makes possible to obtain the inverse translation $^\circ$ as the above figure-1:
Due to space considertations, precise definitions of $+$ and $^\circ$ are omitted.

Theorem (Soundness):
If P: $\Gamma \vdash_L \phi$, then $\Gamma(P)^+ \vdash_{\lambda L} M{:}\phi^+$ for some M.

Proof is by induction on the structure of a proof figure P [3].

Theorem (Completeness):
For every $A_p{:}*_p$ if $\Gamma \vdash_{\lambda L} M{:}A_p$ for some M, then $\Gamma^\circ \vdash_L A_p^\circ$.

Proof is by induction on the structure of a deduction $\Gamma \vdash_{\lambda L} M{:}A_p$ [3].

4. Relationship between L and λL

We summarize the relationship between logic L and type system λL. *W. r. t.* the structure of a logic (figure-2), a term t∈**TERM** in L corresponds to a term M or a type in λL and a sort S∈**SORT** corresponds to a type A_t or a kind:

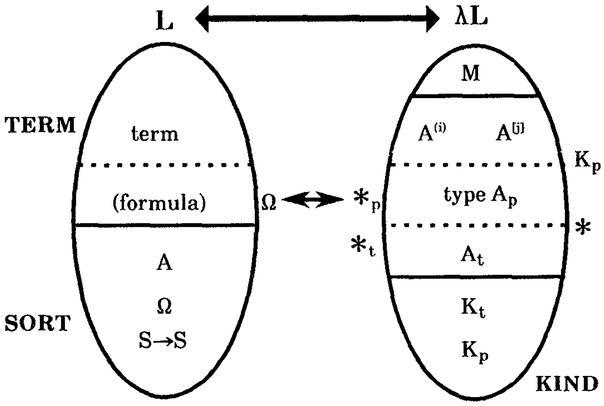


figure-2

W.r.t. rules, inference rules in L correspond to formulation rules of well-formed elements of a type in λL (table-1):

table-1

L	λL
(assumption)	$\dfrac{\vdash \Gamma \quad z{:}A_p \in \Gamma}{\Gamma \vdash z{:}A_p}$
(⊃I)	$(*_p, *_p){:}\ \dfrac{\Gamma \vdash A_p{:}*_p \quad \Gamma, x{:}A_p \vdash M{:}B_p}{\Gamma \vdash \lambda x{:}A_p.M : \Pi x{:}A_p.B_p}$
(∀I) where S∈**SORT-t**	$(*_t, *_p){:}\ \dfrac{\Gamma \vdash A_t{:}*_t \quad \Gamma, x{:}A_t \vdash M{:}B_p}{\Gamma \vdash \lambda x{:}A_t.M : \Pi x{:}A_t.B_p}$
(∀I) where S∈**SORT-p**	$(K_p, *_p){:}\ \dfrac{\Gamma \vdash K_p \quad \Gamma, X_p{:}K_p \vdash M{:}A_p}{\Gamma \vdash \lambda X_p{:}K_p.M : \Pi X_p{:}K_p.A_p}$

W.r.t. deductions, as shown in the diagrams (figure-3, figure-4) the deductions commute with the translation $+$:

figure-3

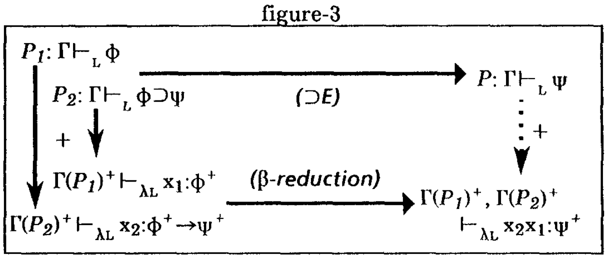
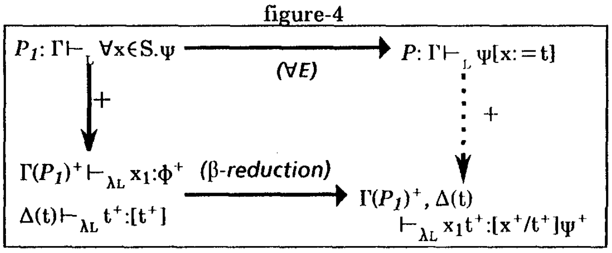


figure-4



5. Conclusion

We have investigated the relationship between logics and type systems. First, we proved the soundness theorem for *intuitionistic higher order many sorted predicate logic* using the notion of formulae-as-types. Next, we proved the completeness theorem for *λPK2* in the restricted class of $*$, i.e., $*_p$. In this sense we can obtain the one-to-one correspondence between Ω and $*_p$. In another words we have shown the correspondence between inference rules and formulation rules of well-formed elements of a type. Hence a formula ϕ can be identified with a type A_p where $A_p{:}*_p$, and we can simulate a deduction P: $\Gamma \vdash_L \phi$ by that of $\Gamma(P)^+ \vdash_{\lambda L} M{:}\phi^+$ for some M. Moreover, since we can construct a context from an assumption with a proof figure, the scope of the translation is wider than the existing one. Therefore whole logical systems can be translated into type systems.

References

[1] Geuvers, H. and Barendregt, H. : *The Interpretation of Logics in Type Systems. Department of Computer Science,* University of Nijmegen, The Netherlands. August 1988.

[2] Howard, W. A. : Hindley, J. R. and.Seldin, J. P. (Eds.) : *The Formulae-as-types Notion of Construction. To H.B.Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism.* Academic Press, 1980 pp.479-490.

[3] Fujita, K., Togashi, A. and Noguchi, S. : *Relationship between logic and type system.* Theoretical Foundations of Computing. COMP 89-18, IEICE Technical Report, pp.11-20.

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