The Convex Configurations of Dissection Puzzles with Seven Pieces

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Abstract: The most famous dissection puzzle is the tangram, which originated in China more than two centuries ago. From around the same time, there is a similar Japanese puzzle called *Sei Shonagon Chie no Ita*. Both are derived by cutting a square of material with straight incisions into seven different-sized pieces, and each piece consists of a few identical right isosceles triangle units. The right isosceles triangle unit is of 1/16 of the square, and the set of 16 units can form 20 different convex polygons. It is known that the tangram can form thirteen convex polygons among 20 convex polygons, and the Sei Shonagon Chie no Ita can form sixteen among them. Therefore, in a sense, the Sei Shonagon Chie no Ita is more expressive than the tangram. Last year, Fox-Epstein and Uehara proposed a more expressive pattern that can form nineteen convex polygons, and show that no set of seven pieces made from sixteen identical right isosceles triangles can form 20. In this paper, we refine their analysis, obtain four expressive patterns that satisfy the condition, and show that these four patterns are all.

Keywords: Dissection puzzles, Sei Shonagon Chie no Ita, Tangram.

1. Introduction

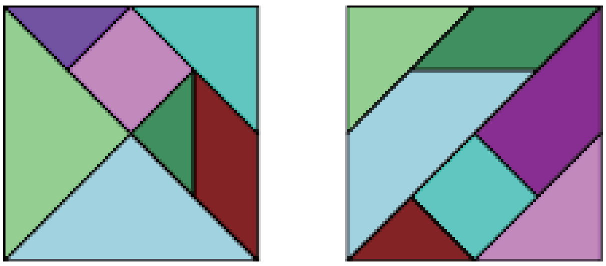

Fig. 1 Left: the tangram in square configuration. Right: Sei Shonagon Chie no Ita pieces in square configuration.

A *dissection puzzle* is a game where, given a set of polygons, one must decide whether they can be placed in the plane in such a way that their union is a target polygon. Rotation and reflection are allowed but scaling and overlapping are not. Formally, a set of polygons S can *form* a polygon P if there is an isomorphism up to rotation and reflection between a partition of P and the polygons of S (i.e. a bijection $f(\cdot)$ from a partition of P to S such that x and $f(x)$ are congruent for all x).

The *tangram* is a set of polygons consisting of a square of material cut by straight incisions into different-sized pieces. See the left diagram in Fig. 1. Of anonymous origin, their first known reference in literature is from 1813 in China [Slo04]. The tangram has grown to be extremely popular throughout the world; now, over 2000 dissection and related puzzles exist for it ([Slo04][Gar87]).

Much less famous is a quite similar Japanese puzzle called *Sei*

Fig. 2 A set of plates in the form of Sei Shonagon Chie no Ita pieces, crafted by Tomomi Takeda in Kanazawa, Japan.

Fig. 3 A typical Sei Shonagon Chie no Ita layout as a square configuration with a hole missing.

Shonagon Chie no Ita. Sei Shonagon was a courtier and famous novelist in Japan, but there is no evidence that the puzzle existed a millennium ago when she was living. *Chie no ita* means *wisdom*

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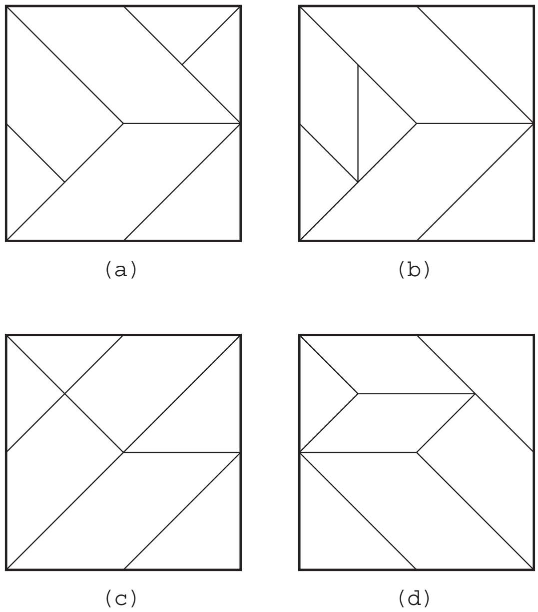

Fig. 5 Four patterns that can form nineteen convex polygons.

plates, which refers to this type of physical puzzle. It is said that the puzzle is named after Sei Shonagon's wisdom. Historically, the Sei Shonagon Chie no Ita first appeared in literature in 1742 [Slo04]. Even in Japan, the tangram is more popular than Sei Shonagon Chie no Ita, though Sei Shonagon Chie no Ita is common enough to have been made into ceramic dinner plates (see e.g. Fig. 2, [Tak14]), and in puzzle communities, it is admired for being able to form some more interesting shapes that the tangram cannot, such as a square configuration with a hole missing (Fig. 3).

Wang and Hsiung considered the number of possible convex (filled) polygons formed by the tangram [WH42]. They first noted that, given sixteen identical isosceles right triangles, one can create the tangram pieces by gluing some edges together. So, clearly, the set of convex polygons one can create from the tangram is a subset of those that sixteen identical isosceles right triangles can form. Embedded in the proof of their main theorem, Wang and Hsiung [WH42] demonstrate that sixteen identical isosceles right triangles can form exactly 20 convex polygons. These 20 are illustrated in Fig. 4. The tangram can realize thirteen of those 20.

It is quite natural to ask how many of these twenty convex polygons the Sei Shonagon Chie no Ita pieces can form. Fox-Epstein and Uehara showed that Sei Shonagon Chie no Ita achieves sixteen convex polygons out of twenty [FEU14][*1]. Therefore, in a sense, we can conclude Sei Shonagon Chie no Ita is more expressive than the tangram: while both the tangram and Sei Shonagon Chie no Ita contain seven pieces made from sixteen identical isosceles right triangles, Sei Shonagon Chie no Ita can form more convex polygons than the tangram. One might next wonder if this can be improved with different shapes. Fox-Epstein and Uehara also show a set of seven pieces that can form nineteen convex polygons among twenty candidates, and that to realize all twenty convex polygons, it is necessary and sufficient to have eleven shapes [FEU14]. In this paper, we refine their analysis, and state that there are four patterns that satisfy the condition (Fig. 5); that is, each of four pattern can form nineteen convex polygons among twenty candidates, and there are no other pattern that has the property.

[*1] Later, we found out that this fact is a folklore in the puzzle society in Japan.

2. Preliminaries

We first notice that the pieces of the Sei Shonagon Chie no Ita can be decomposed into sixteen identical right isosceles triangles, just like the tangram.

We make use of two important results from Wang and Hsiung [WH42]. First, there are only 20 candidate convex polygons that we need to consider (Fig. 4). Second, in any convex polygon they can form, the bases of the sixteen triangles can be pairwise colinear, parallel, or perpendicular ([WH42], Lemma 1). This means we only need to consider configurations that could be embedded with triangle and target polygon vertices on integer coordinates.

Hereafter, one of the sixteen identical right isosceles triangles is called a *tile* for short. As shown in [FEU14], there is a set of seven polygons composed from sixteen tiles that can form nineteen distinct convex polygons. Furthermore, no set of seven polygons composed of sixteen tiles can form 20 distinct convex polygons. Hereafter, we enumerate four possible sets of seven polygons composed from sixteen tiles that can form nineteen distinct convex polygons. This is shown by a case analysis, hence it is easy to follow that there is no other set.

3. Optimal seven piece puzzles

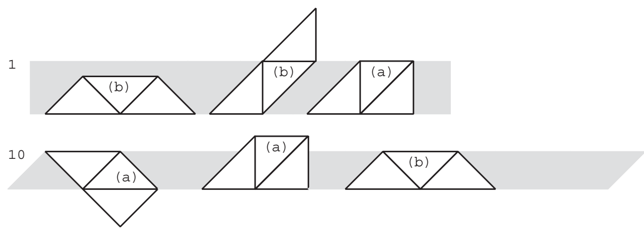

Fig. 6 Two skinny shapes and piece of three tiles.

Our first lemma states that we can fix the nineteen convex polygons out of twenty that can be filled by our puzzle.

Lemma 1 Any set of seven pieces composed from sixteen tiles can fill nineteen of twenty convex polygons except the convex shape 10 in Fig. 4.

Proof. We first observe that the average number of tiles in a piece is $16/7 = 2.285\cdots$. Therefore, any dissection pattern contains at least one piece containing at least three tiles. Then, there are two possible pieces that consists of three tiles (a) and (b) as shown in Fig. 6. If we choose (a), we cannot fill the polygon 10. On the other hand, if we choose (b), we cannot fill the polygon 1. However, when we omit the polygon 1, we also have to omit the polygons 2 and 3 in Fig. 4. Therefore, to fill nineteen of them, we have to omit 10. □

In the proof of Lemma 1, we choose polygons 1, 2, and 3 and omit the polygon 10. Then we can also say that any piece containing at least three tiles should be extended from the tile (a) in Fig. 6, and we cannot use the tile (b) and its extensions. To fill the

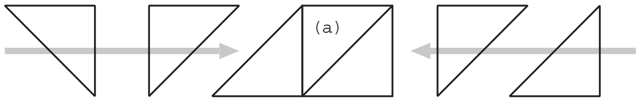

Fig. 7 Extensions of piece of size three.

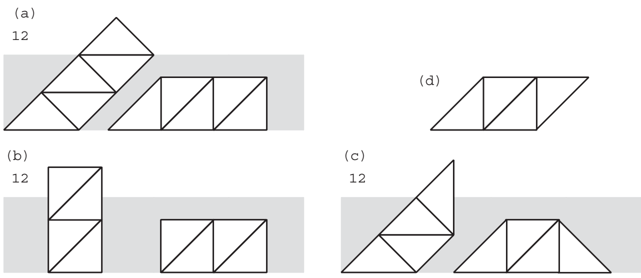

Fig. 8 Pieces of four/five tiles that cannot fill the polygon 12.

shape 1, the possible tiles of size at least three are given in Fig. 7. However, if the number of tiles is greater than 4, it cannot fill the polygon 12 (Fig. 8(a)). Moreover, even if the number of tiles is 4, two of three possible pieces cannot fill the polygon 12, either (Fig. 8(b)(c)). Therefore, the only possible piece of four tiles is one in Fig. 8(d). From above discussion, now we can summary the possible pieces below:

Lemma 2 Any set of seven pieces that can fill nineteen convex polygons should contain the elements in Fig. 9.

We name each tile as t4, t3, t2-1, t2-2, t2-3, and t1 as in the figure. We also call any of t2-1, t2-2, and t2-3 *tile t2* if we do not need to distinguish them.

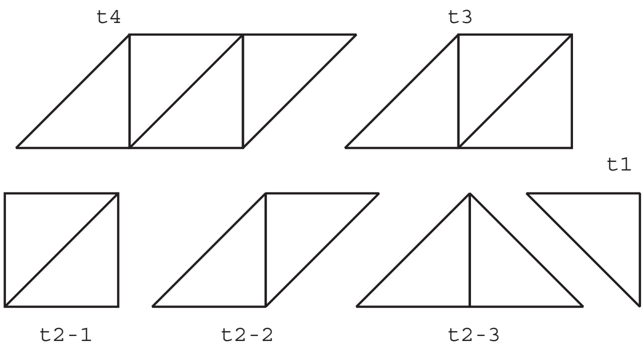

Fig. 9 Possible pieces to make nineteen convex polygons by seven pieces.

Let v, w, x, y be the number of tiles t4, t3, t2, and t1, respectively. Then they are integers, and we have $v + w + x + y = 7$ and $4v + 3w + 2x + y = 16$. The conditions are satisfied only when $(v, w, x, y) = (3, 0, 0, 4), (2, 1, 1, 3), (2, 0, 3, 2), (1, 3, 0, 3), (1, 2, 2, 2), (1, 1, 4, 1), (1, 0, 6, 0), (0, 4, 1, 2), (0, 3, 3, 1), (0, 2, 5, 0)$. For each of (v, w, x, y) for $x > 0$, we have three kinds of t2 tiles. Considering the combinations, we obtain 98 candidates of the sets. We analyze one by one by our hands:

(3,0,0,4),(1,3,0,3)

We have one set in each case, and it cannot fill the convex polygon 14 (or square).

(2,1,1,3)

We have three sets. When we choose t2-3, we have the set (a) in Fig. 5. Using the other two, we cannot fill the square.

(2,0,3,2)

We have ten combinations for t2 tiles. Among them, we can find the set (b) in Fig. 5. Two sets cannot fill the convex polygon 17, and the other sets cannot fill the square.

(1,2,2,2)

We have six sets, and one is the set (c) in Fig. 5. The other sets cannot fill the square.

(1,1,4,1)

We have 15 sets, and one of them is the set (d) in Fig. 5. The other sets cannot fill either the convex polygons 17, 19, or the square.

(1,0,6,0)

We have 28 sets, but none of them can fill the polygon 1.

(0,4,1,2),(0,3,3,1),(0,2,5,0)

We have 34 sets in total, but none of them can fill the polygon 19 or the square.

Therefore, we conclude that there are four possible sets that can fill nineteen out of twenty convex polygons shown in Fig. 4.

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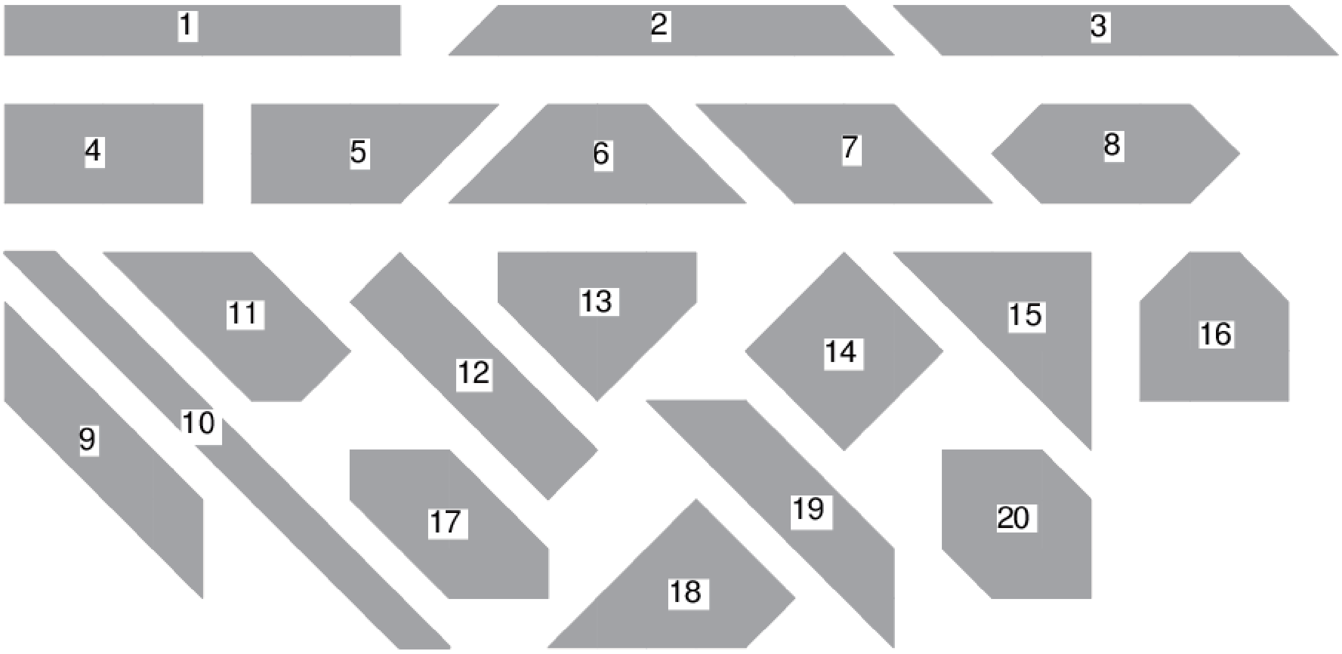

Fig. 4 All 20 potential convex polygons.