Improved fixed parameter algorithm for two-layer crossing minimization

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Abstract: We give an algorithm that decides whether the bipartite crossing number of a given graph is at most k. The running time of the algorithm is $2^{O(k)}n^{O(1)}$, where n is the number of vertices of the input graph, which improves the previous algorithm due to Kobayashi *et al.* (TCS 2014) that runs in $2^{O(k\log k)}n^{O(1)}$ time. This result is based on a combinarotial upper bound on the number of two-layer drawings of a connected bipartite graph with a bounded crossing number.

1. Introduction

A *two-layer drawing* of a bipartite graph is a drawing in which the vertices in one color class are placed on a straight line, the vertices in another color class are placed on another straight line parallel to the first line, and each edge is drawn as a straight line segment. The two parallel lines are called *layers*. A *crossing* in a two-layer drawing is a pair of edges that intersect each other at a point distinct from the two straight lines placing vertices. The problem of finding a two-layer drawing with the minimum number of crossings, called *two-layer crossing minimization* (or simply TLCM), is dealt with as a combinatorial problem: the number of crossings in a two-layer drawing is determined by the order of vertices on each layer.

TLCM is shown to be NP-hard [2] and is solvable in polynomial time for trees [6] and bipartite permutation graphs [7]. The original proof in [2] shows, in fact, the hardness of TLCM for multigraphs, however, recently [5] shows the hardness of TLCM for simple graphs. TLCM has been studied from a parameterized perspective. In this context, we are asked if there is a two-layer drawing of a given bipartite graph with at most k crossings, where k is the parameter for the paramerized problem. Dujimović *et al.* [1] first show that this parameteried problem is fixed parameter tractable. More precisely, they give an algorithm that decides whether a given graph (not necessary bipartite) with n vertices has an h-layer drawing with at most k crossings in $2^{O((h+k)^3)}n$ time (see [1], for details). The current and two other authors [4] improve the running time for the restricted case $h = 2$, namely TLCM, to $2^{O(k\log k)} + n^{O(1)}$. Moreover they, for the first time, show that TLCM admits a polynomial kernelization. In this paper, we give a faster fixed parameter algorithm for TLCM.

Theorem 1. *There is an algorithm that decides whether a given bipartite graph has a two-layer drawing with at most k crossings whose running time is* $2^{O(k)} + n^{O(1)}$*, where n is the number of vertices of the input graph.*

To establish Theorem 1, we analyze the number of two-layer drawings whose crossing number is at most k and enumerate all such drawings in the claimed running time. This strategy is inspired by the work of Gutin *et al.* [3]. They consider a parameterzed version of the linear arrangement problem and give a fixed paramter algorithm for the problem. To this end, they analyze the number of feasible solutions of the problem for a spanning tree of the input graph. However, this analysis can not be applied to TLCM. See Lemma 1.

2. Preliminaries

Let G be a bipartite graph with a prescribed bipartition, denoted by $(X(G), Y(G))$, of the vertex set. We denote by $E(G) \subseteq X(G) \times Y(G)$ the set of edges of G. We call a vertex of degree one a *leaf* and the edge incident to the leaf a *leaf edge*. An edge that is not leaf edge is called a *non-leaf edge*.

For a set S, a layout on S is a bijection f from S to $\{1, 2, \ldots, |S|\}$. A *two-layer drawing* D of G is a triple (G, f_X, f_Y), where f_X and f_Y are layouts on $X(G)$ and $Y(G)$, respectively. The *bipartite crossing number* of D, denoted by $\mathrm{bcr}(D)$, is

$$\sum_{(x,y)\in E(G)} |\{(x', y') \in E(G) : f_X(x) < f_X(x'), f_Y(y') < f_Y(y)\}|.$$

The *bipartite crossing number* $\mathrm{bcr}(G)$ of G is the minimum k such that there is a two-layer drawing D of G with $\mathrm{bcr}(D) = k$.

3. Combinatorial upper bound

In this section, fix a connected bipartite graph G and an integer k, we give an upper bound on the number of two-layer drawings of G with at most k crossings. Let $n = |X(G) \cup Y(G)|$. One may notice that a trivial upper bound $(n-1)!$ is essentially tight since $K_{1,n-1}$ has $(n-1)!$ different two-layer drawings without any crossings. One would, however, also notice that all of those drawings

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are equivalent in a natural sense. More generally, it is straightforard to verify that, for every bipartite graph G, there is an optimal two-layer drawing of G in which all the leaves adajacent to v appear consecutively in their layer for each non-leaf vertex v. Therfore, we may treat such sibling leaves as a single leaf, weighting the corresponding leaf edge by the number of represented leaves [4]. We call a pair of leaves a *sibling pair*, if they have a common neighbor.

Lemma 1. *Suppose G has no sibling pairs. Then, the number of two-layer drawings of G with at most k crossings is* $2^{O(n+k)}$.

Proof. The lemma is trivial when $|X(G)| = 1$ or $|Y(G)| = 1$. Hence we assume otherwise. Choose $r \in X(G)$ arbitrarily and define a function $T : X(G) \setminus \{r\} \to X(G)$ that satisfies the following conditions:

(1) for each $x \in X(G) \setminus \{r\}$, there is a path P_x of length two between x and $T(x)$ in G,
(2) for each $e \in E(G)$, $|\{x \in X(G) \setminus \{r\} : e \in E(P_x)\}| \leq 2$, and
(3) $(X(G), \{\{x, T(x)\} : x \in X(G) \setminus \{r\}\})$ forms a tree.

The function T is defined as follows. Let H be a bipartite multigraph obtained from G by replacing each edge by two parallel edges. Since every vertex in H has even degree and H is connected, H has an eulerian tour. Then, we fix an eulerien tour starting at r. For $x \in X(G) \setminus \{r\}$, we let $T(x)$ be the vertex in $X(G)$ that is visited immediately after the last visiting of x in the tour. It is easy to see that the function T satisfies condition 1 and 2. Since $(X(G), \{\{x, T(x)\} : x \in X(G) \setminus \{r\}\})$ has $|X(G)| - 1$ edges and, by the construction of T, has no cycles, condition 3 holds. Fix a two-layer drawing $D = (G, f_X, f_Y)$. For $x \in X(G) \setminus \{r\}$, we define $g(x) = |f_X(x) - f_X(T(x))| - 1$. In words, $g(x)$ is the number of vertices of $X(G)$ that lie between x and $T(x)$ in D. Since G has no sibling pairs, there is at most one leaf in $X(G)$ adjacent to $y \in V(P_x) \cap Y(G)$. Observe that each edge incident to a vertex counted by $g(x)$ except for such a leaf (if it exists) makes a crossing with an edge of P_x. Considering double counts and the fact that each edge belongs to at most two paths P_x, $x \in X(G)$, we have

$$\frac{1}{4} \sum_{x \in X(G) \setminus \{r\}} (g(x) - 1) \leq k.$$

Therefore, the number of possible functions g is at most

$$\binom{4k + 2n - 3}{4k + n - 1} \leq 2^{4k+2n-3}.$$

By condition 3, f_X is determined by the values of $g(x)$ and the signs of $|f_X(x) - f_X(T(x))|$ for each $x \in X(G) \setminus \{r\}$ and the value of $f_X(r)$. Hence the number of possible layouts f_X is bounded by $2^{4k+2n-3} \cdot 2^{n-1} \cdot n$. Applying the same argument to $Y(G)$ proves the lemma. □

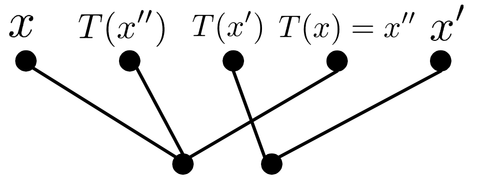

Fig. 1 An example of paths P_x, $P_{x'}$, and $P_{x''}$. The crossing is counted by $g(x)$, $g(x')$, and $g(x'')$.

The above proof immediately gives an algorithm that enumerates all two-layer drawings of G with at most k crossings in $2^{O(n+k)}$ time provided G is connected and has no sibling pairs.

4. FPT algorithm

In this section, we will design a fixed parameter algorithm for TLCM using the enumeration algorithm in the previous section. Our fixed parameter algorithm uses a kernelization result due to [4]. First, we define a slightly general problem of TLCM as follows. Consider a two-layer drawing of an edge weighted bipartite graph. We assume that, in this paper, each edge has weight at least one. The *weight* of a crossing is the product of the weights of the crossing edges. The crossing number of the drawing is the sum of the weights of all crossings in the drawing. The bipartite crossing number of an edge weighted bipartite graph is defined analogously. A *leaf edge weighted graph* is an edge weighted graph where each non-leaf edge has weight exactly one. [4] gives in fact a kernel for TLCM for leaf edge weighted bipartite graphs.

Theorem 2 ([4])**.** *There is a polynomial time algorithm that, given a connected bipartite graph G and an integer k, computes a leaf edge weighted connected bipartite graph H such that H has no sibling pairs,* $|E(H)| = O(k)$, *and* $\mathrm{bcr}(G) \leq k$ *if and only if* $\mathrm{bcr}(H) \leq k$.

Given a bipartite graph G and an integer k, we apply the algorithm in Theorem 2 to each connected component of G. For each output, by Lemma 1, the number of two-layer drawings with at most k crossings is $2^{O(k)}$. If one of the outputs has no such drawings, the answer is negative. The bipartite crossing number of each output can be computed in $2^{O(k)}$ time by using the enumeration algorithm in the previous section and hence Theorem 1 holds.

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References

[1] V. Dujmović, M.R. Fellows, M. Kitching, G. Liotta, C. McCartin, N. Nishimura, P. Ragde, F. Rosamond, M. Suderman, S. Whitesides, D.R. Wood. On the parameterized complexity of layered graph drawing. Algorithmica 52(2), 267–292, 2008.
[2] M.R. Garey, D.S. Johnson, Crossing number is NP-complete. SIAM Journal on Algebraic and Discrete Methods (4)3, 312–316, 1982.
[3] G. Gutin, A. Rafiey, S. Szeider, A. Yeo, The linear arrangement problem parameterized above guaranteed value. Theory of Computing Systems 41(3), 521–538, 2007.
[4] Y. Kobayashi, H. Maruta, Y. Nakae, H. Tamaki, A linear edge kernel for two-layer crossing minimization. Theoretical Computer Science 554(16), 74–81, 2014.
[5] M. Schaefer, The graph crossing number and its variant: a survey. The Electronic Journal of Combinatorics 20(2), DS21, 2013.
[6] F. Shahrokhi, O. Sýkora, L.A.Székely, I. Vrt'o, On bipartite drawings and the linear arrangement problem. SIAM Journal on Computing 30(6), 1773–1789, 2001.
[7] J. Spinrad, A. Brandstädt, L. Stewart, Bipartite permutation graphs. Discrete Applied Mathematics 18(3), 279–292, 1987.