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Designing Various Multivariate Analysis at Will via Generalized Pairwise Expression*

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Abstract: It is well known that dimensionality reduction based on multivariate analysis methods and their kernelized extensions can be formulated as generalized eigenvalue problems of scatter matrices, Gram matrices or their augmented matrices. This paper provides a generic and theoretical framework of multivariate analysis introducing a new expression for scatter matrices and Gram matrices, called Generalized Pairwise Expression (GPE). This expression is quite compact but highly powerful. The framework includes not only (1) the traditional multivariate analysis methods but also (2) several regularization techniques, (3) localization techniques, (4) clustering methods based on generalized eigenvalue problems, and (5) their semi-supervised extensions. This paper also presents a methodology for designing a desired multivariate analysis method from the proposed framework. The methodology is quite simple: adopting the above mentioned special cases as templates, and generating a new method by combining these templates appropriately. Through this methodology, we can freely design various tailor-made methods for specific purposes or domains.

Keywords: multivariate analysis, dimensionality reduction, generalized eigenvalue problem, pairwise expression, kernel method, clustering, semi-supervised learning, regularization

1. Introduction

We can easily obtain a massive collection of texts (long articles1, microblogs2), images1,2,3, videos5,6 and musics7 nowadays. However, we are now facing a difficulty in finding an intrinsic trend and nature of such a massive collection of data. Multivariate analysis8 is traditional, quite simple but might be one of the powerful tools to obtain a hidden structure embedded in the data, via classification, regression and clustering9,10. Actually, multivariate analysis has been still an important tool, and recent reports showed its effectiveness for several applications, e.g., human detection11, image annotation12,13, and sensor data mining14,15,16,17.

Principal component analysis (PCA)18, Fisher discriminant analysis (FDA)19, multivariate linear regression (MLR), canonical correlation analysis (CCA)18, and partial least squares (PLS)20 are well known as standard multivariate analysis methods. These methods can be formulated as a generalized eigenvalue problem of a scatter matrix or an augmented matrix composed of several scatter matrices. Several extended researches tried to tackle the small sample size problem21, i.e., the situation where the number of training samples is small compared with their dimensionality (e.g., robust PCA22,23,24,25 and robust FDA26,27,28). Kernelized extensions of those standard methods have been also developed to deal with non-vector samples and non-linear analysis (e.g., kernel PCA29, kernel FDA30,31,32, kernel MLR33, and kernel CCA34,35). They can be formulated as a generalized eigenvalue problem of an augmented matrix composed of Gram matrices, instead of scatter matrices. Kernel multivariate analysis often needs some regularization techniques such as \( \ell_2 \)-norm regularization36,37,38 to inhibit overfitting and the graph Laplacian method39 to fit underlying data manifolds smoothly. In addition, improvements of robustness against outliers and non-Gaussianity (i.e., multi-dimensional scaling (MDS)40, locality preserving projection (LPP)41 and local Fisher discriminant analysis (LFDA)42) and their extensions to semi-supervised dimensionality reduction39,43,44 have been considered.

In addition, a lot of multivariate analysis methods and several trials to unify these methods have been presented so far. Borga et al.45 and De Bie et al.46 showed that several major linear multivariate analysis method can be formulated by a unified form of generalized eigenvalue problems by introducing the augmented matrix expression. Sun et al.47,48 showed the equivalence between a certain class of generalized eigenvalue problems and least squares problems under a mild assumption. De la Torre49,50 further extended their work to a various kind of component analysis methods by introducing the formulation of least-squares weighted kernel reduced rank regression (LS-WKRRR). However, freely designing a tailor-made multi-

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multivariate analysis for a specific purpose or domain still remains an open problem. Until now, researchers and engineers have had to choose one of the existing methods that seems best to address the problem of interest, or had to laboriously develop a new analysis method tailored specifically for that purpose.

In view of the above discussions, this paper provides a new expression of second-order statistics including covariance matrices and Gram matrices, which we call the Generalized Pairwise Expression (GPE). GPE is originated in Pairwise Expression (PE)[42], [43], [51] that tries to describe the relation between pairs of samples regardless whether pairs are close together or far apart. Through the PE framework, we can obtain an interpretation of a multivariate analysis method as to how it works. Our GPE framework extends the PE framework to larger classes of multivariate analysis methods, and it can be also diverted to designing a new multivariate analysis method. Our main contributions of this paper can be summarized as follows:

(1) GPE makes it easy to design a new multivariate analysis method with desired properties without any special knowledge of multivariate analysis.

(2) The methodology is quite simple: Exploiting the above mentioned existing methods as templates, and constructing a new method by combining these templates appropriately.

(3) It is also possible to individually select and arrange samples for calculating the second-order statistics of the methods to be combined, which enables us to extend multivariate analysis methods to semi-supervised ones and multi-modal ones, where some parts are calculated from only labeled samples, and the other parts are obtained from both labeled and unlabeled samples.

The rest of this paper is organized as follows: Section 2 defines a class of multivariate analysis methods we are concerned with in this paper. Next, Section 3 describes our proposed framework, GPE, and its fundamental properties. These properties provide a methodology to design multivariate analysis methods with desired characteristics. Then, Section 4 reviews major multivariate analysis methods from the viewpoint of GPE. This review will give us templates of the GPEs for designing desired methods. After the above preparations, Section 5 demonstrates how to design a new multivariate analysis method. By replicating the methodology shown in the preceding sections, we can easily design various multivariate analysis methods at will. Additionally, Section 6 describes a non-linear and/or non-vector extension of GPE with the help of the kernel trick, which is not trivial. With this extension, non-linear dimensionality reduction, and several clustering methods are all in the class of multivariate analysis methods we are concerned with.

2. Multivariate Analysis for Vector Data

2.1 Preliminaries

Consider two sets $X$ and $Y$ of samples $\{x_i, y_i\}$, where each set contains $N_x$ and $N_y$ samples, and each sample can be expressed as a vector with $d_x$ and $d_y$ dimensions, respectively, as follows:

\[
X = \{x_1, \ldots, x_{N_x}\},
Y = \{y_1, \ldots, y_{N_y}\}(N \leq N_x).
\]

For brevity, both of the sample sets $X$ and $Y$ are supposed to be centered on the origin by subtracting the mean from each component. Suppose that samples $x_1$, $y_1$, with the same suffix are co-occurring. Each set $X$ and $Y$ of samples is separated into the following two types: Complete sample sets $X^{(C)}$ and $Y^{(C)}$ so that every sample $x_n$ (resp. $y_n$) has co-occurring sample $y_n$ (resp. $x_n$), and incomplete sample sets $X^{(I)}$ and $Y^{(I)}$ so that every sample $x_n$ (resp. $y_n$) cannot find the co-occurring sample.

\[
X^{(C)} = \{x_1, x_2, \ldots, x_{N_x}\},
Y^{(C)} = \{y_1, y_2, \ldots, y_{N_y}\},
X^{(I)} = \{x_{N_x+1}, x_{N_x+2}, \ldots, x_{N_x+N_y}\},
Y^{(I)} = \{y_{N_y+1}, y_{N_y+2}, \ldots, y_{N_x+N_y}\}.
\]

First, we concentrate on the case that $N_x = N_y = N$, namely all the samples are paired, unless otherwise stated.

2.2 Formulation

Many linear multivariate analysis methods developed so far involve an optimization problem of the following form for a $d$-dimensional vector $w$:

\[
w^{(opt)} = \arg \max_{w \in \mathbb{R}^d} R(w),
\]

\[
R(w) = w^\top \tilde{C}w(w^\top \tilde{C}w)^{-1},
\]

where $\tilde{C}$ and $\tilde{C}$ are square matrices with certain statistical nature. For example, $\tilde{C}$ is a scatter matrix of $X$ and $\tilde{C}$ is an identity matrix in PCA, and $\tilde{C}$ is a between-class scatter matrix and $\tilde{C}$ is a within-class scatter matrix in FDA. Roughly speaking, $\tilde{C}$ encodes the quantity that we want to increase, and $\tilde{C}$ corresponds to the quantity that we want to decrease. The denominator of the function $R(w)$ is often normalized to remove scale ambiguity, resulting in the following form:

\[
w^{(opt)} = \arg \max_{w \in \mathbb{R}^d} R_1(w) \text{ s.t. } R_2(w) = 1,
\]

\[
R_1(w) = w^\top \tilde{C}w, \quad R_2(w) = w^\top \tilde{C}w.
\]

The above optimization problem can be converted to the following generalized eigenvalue problem via the Lagrange multiplier method:

\[
\tilde{C}w = \lambda \tilde{C}w.
\]

The solution $w_k (k = 1, 2, \ldots, r)$ of the above generalized eigenvalue problem gives a solution of the original multivariate analysis formulated in Eq. (1).

It can be confirmed that Eq. (1) is invariant against any kinds of linear transformations, i.e., a vector $Uw^{(opt)}$ transformed by any
The above expression is called the pairwise expression (PE) of the second-order statistics $S_{Q,xx}$. If $Q$ is a weight matrix for a graph with $N$ nodes, $L_Q$ can be regarded as a graph Laplacian matrix in the spectral graph theory. If $Q$ is symmetric and its elements are all non-negative, $L_Q$ is known to be positive semi-definite.

Here, we extend PE to the following expression introducing an additional matrix independent of $Q$:

$$S_{Q,xx} = XL_QX^T + L_2,$$

where $L_{Q,1}$ is an $N \times N$ positive semi-definite matrix, and $L_2$ is a $d_i \times d_i$ non-negative symmetric matrix. We do not have to explicitly consider the matrix $Q$ for the following discussions:

$$S_{xx} = XL_2X^T + L_2. \tag{6}$$

After all, we call this expression as the generalized pairwise expression (GPE) (See Fig. 1). The first term of Eq. (6) is called the data term since it depends on the sample data, and the second term is called the bias term.

### 3.2 Properties

We derive the following fundamental properties of GPE from the definition, if the number of samples, $N$, is sufficiently large:

1. If $A$ is GPE and $\beta > 0$ is a constant, then $\beta A$ is also GPE.
2. If both $A$ and $B$ are GPE, then $A + B$ is also GPE.
3. If both $A$ and $B$ are GPE, then $AB$ is also GPE.

**Proof.** The first and second claims can be easily proved, so we concentrate on proving the third one.

First, let us denote $A$ and $B$ as follows:

$$A = XL_{A1}X^T + L_{A2},$$
$$B = YL_{B1}X^T + L_{B2},$$

where $L_{A1}$ and $L_{B1}$ are positive semi-definite $N \times N$ matrices, and $L_{A2}$ and $L_{B2}$ are $d_i \times d_i$ non-negative symmetric matrices. Then, we obtain

$$AB = (XL_{A1}X^T + L_{A2})(XL_{B1}X^T + L_{B2}),$$

$$= XL_{A1}X^T XL_{B1}X^T + XL_{A1}X^T L_{B2} + L_{A2}L_{B2}.$$

We can find some matrices $L_{ci}$ ($i = 1, 2, 3$) satisfying the following relationships, if $N \geq d_i$:

$$L_{c1} = L_{A1}X^T XL_{B1},$$
$$L_{c2}X^T = L_{A1}X^T L_{B2},$$
$$XL_{c3} = L_{A2}XL_{B1}.$$

Here, we will show that those matrices $L_{ci}$ ($i = 1, 2, 3$) are all positive semi-definite. For any matrix $X \in \mathbb{R}^{d_i \times N}$, we have

$$XL_{c1}X^T = (XL_{A1}X^T)(XL_{B1}X^T),$$
$$XL_{c2}X^T = (XL_{A1}X^T)L_{B2},$$
$$XL_{c3}X^T = L_{A2}(XL_{B1}X^T).$$

Recalling that $L_{A1}$ and $L_{B1}$ are positive semi-definite and $L_{A2}$ and $L_{B2}$ are non-negative symmetric, we can see that all the above matrix product $XL_{ci}X^T$ ($i = 1, 2, 3$) are non-negative, which means that the matrices $L_{ci}$ ($i = 1, 2, 3$) are all positive semi-definite. This implies that

$$AB = X(L_{c1} + L_{c2} + L_{c3})X^T + L_{A2}L_{B2}.$$
\[ XL_{D1}X^T + L_{D2}, \]

for a positive semi-definite matrix \( L_{D1} \) and a non-negative symmetric matrix \( L_{D2} \), which means \( AB \) is also GPE.

These fundamental properties of GPE provide us a promising way to design various multivariate analysis methods very easily, namely with addition, weighting and multiplication of GPEs of existing methods with desired characteristics. The rest of the problem is to reveal GPE of existing methods and the function of every type of combinations (addition and/or multiplication), which will be described in the next section.

4. Reviewing Multivariate Analysis

4.1 Preliminaries

This section reviews major multivariate analysis methods from the viewpoint of GPE. As shown in Sections 2 and 3, the GPEs of PCA and FDA respectively are given by

\[
\tilde{C}_{\text{PCA}}^{(PCA)} = S_{xx}, \quad \tilde{C}_{\text{FDA}}^{(FDA)} = S_{xx}^{(w)},
\]

where \( S_{xx}^{(b)} \) and \( S_{xx}^{(w)} \) are respectively between-class and within-class scatter matrices of \( X \). From these examples, a scatter matrix \( S_{xx} \) is a typical example of the data term in GPE, and an identity matrix \( I_d \) is a typical example of the bias term. Note that unlike FDA, \( \tilde{C}_{\text{PCA}} \) does not have a PE since \( I_d \) cannot be expressed in a pairwise form. This indicates the significance of introducing GPE when reviewing various multivariate analysis methods within a unified framework.

4.2 Canonical Correlation Analysis (CCA)

Canonical correlation analysis (CCA)[18] is a method of correlating linear relationships between two sample sets. Formally, CCA finds a new coordinate \((w_x, w_y)\) to maximize the correlation between the two vectors in the new coordinates. In other words, the function \( \rho(w_x, w_y|X, Y) \) to be maximized is

\[
\rho^{(CCA)}(w_x, w_y|X, Y) = \frac{\langle X^Tw_x, Y^Tw_y \rangle}{\|X^Tw_x\| \|Y^Tw_y\|} = \frac{\mathbb{E}[\langle w_x, x \rangle \langle w_y, y \rangle]}{\sqrt{\mathbb{E}[\langle w_x, x \rangle^2] \mathbb{E}[\langle w_y, y \rangle^2]}} = \frac{w_y^T \mathbb{E}[x y^T] w_x}{\sqrt{w_y^T \mathbb{E}[x x^T] w_x w_y^T \mathbb{E}[y y^T] w_y}} = \frac{w_y^T S_{xy} w_x}{\sqrt{w_y^T S_{xx} w_x w_y^T S_{yy} w_y}}, \tag{7}
\]

where \( \mathbb{E}[\cdot] \) denotes an empirical expectation. The maximum of the function \( \rho(X^{(C)}, Y^{(C)}) \) is not affected by re-scaling \( w_x \) and \( w_y \) either together or independently. Therefore, the maximization of \( \rho(X^{(C)}, Y^{(C)}) \) is equivalent to maximizing the numerator of \( \rho(X^{(C)}, Y^{(C)}) \) subject to \( w_x^T S_{xx} w_x = w_y^T S_{yy} w_y = 1 \). Taking derivatives of the corresponding Lagrangian with respect to \( w_x \) and \( w_y \), we obtain

\[
S_{xy} w_y - \lambda S_{xx} w_x = 0, \quad S_{yx} w_x - \lambda S_{yy} w_y = 0,
\]

where \( \lambda \) is a Lagrange multiplier.

From the above discussion, the GPE of CCA can be obtained as follows:

\[
\tilde{C}_{\text{CCA}} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix}, \quad \tilde{C}_{\text{FDA}} = \begin{pmatrix} S_{xx} & 0 \\ 0 & S_{yy} \end{pmatrix},
\]

\[ w = (w_x^T, w_y^T)^T. \]

We additionally note that when every sample \( y_m \) in \( Y \) represents a class indicator vectors, namely \( y_m \in \{0, 1\}^M, \sum_{m=1}^M y_m = 1 \) and \( M \) is the number of classes, CCA is reduced to FDA [53]6. Thus, CCA can be regarded as a generalized variant of FDA so that each sample can belong to multiple classes.

4.3 Multiple Linear Regression (MLR)

Multiple linear regression (MLR) is a method of finding a projection matrix \( W \) with the minimum squared error between \( y \) and its linear approximation \( Wx \). For simplicity, we first consider the case that the projection matrix \( W \) is with rank 1, which can be written as a direct product of two bases \( w_x \) and \( w_y \). This assumption is useful to understand MLR from the viewpoint of GPE. Then, the objective function to be minimized is the following squared error:

\[
e^{(MLR)}(w_x, w_y|X, Y) = \mathbb{E}[\|y - \alpha w_x w_y^T x\|^2]
\]

\[
= \mathbb{E}[y^T y - 2\alpha w_y^T \mathbb{E}[y^T x] w_x + \alpha^2 w_y^T \mathbb{E}[x x^T] w_x]
\]

\[
= \mathbb{E}[y^T y] - 2\alpha w_y^T S_{xx} w_x + \alpha^2 w_y^T S_{xx} w_x.
\]

To get an expression for \( \alpha \), we calculate the derivative

\[
\frac{\partial}{\partial \alpha} e^{(MLR)}(w_x, w_y|X, Y) = 2(\alpha w_y^T S_{xx} w_x - w_y^T S_{yy} w_y) = 0,
\]

which gives

\[
\alpha = (w_y^T S_{xx} w_x)^{-1}. \quad \tag{8}
\]

Then, we obtain

\[
e^{(MLR)}(w_x, w_y|X, Y) = \mathbb{E}[y^T y] - (w_y^T S_{xx} w_x)^{-1} w_y^T S_{yy} w_y.
\]

Since the squared error cannot be negative and the first term of the objective function is independent of the two directions \( w_x \) and \( w_y \), we can minimize it by maximizing the following generalized Rayleigh quotient:

\[
\rho^{(MLR)}(w_x, w_y|X, Y) = \frac{w_y^T S_{xy} w_x}{w_y^T S_{xx} w_x w_y^T S_{yy} w_y}.
\]

Note that the technical report [53] includes several mistakes in the discussion as to the equivalence between CCA and FDA.
where $w_x$ and $w_y$ are supposed to be normalized as $w_x^T S_x w_x = 1$ and $w_y^T w_y = 1$. By comparing the above equation and Eq. (7) and the objective function for CCA, we can see that MLR is a special case of CCA, and

$$
\hat{E}^{(MLR)} = \begin{pmatrix} \mathbf{0} & S_{xy} \\ S_{yx} & 0 \end{pmatrix},
\hat{C}^{(MLR)} = \begin{pmatrix} S_{xx} & 0 \\ 0 & I_{d_y} \end{pmatrix},
$$

$$
w = (w_x^T, w_y^T)^T.
$$

The above derivation shows a part of the equivalence between the generalized eigenproblem and the least squares, which have been already revealed by Sun et al. [47], [48]. This equivalence property will be often exploited in the following discussions.

### 4.4 Principal Component Regression (PCR)

Principal component regression (PCR) [54] is a variant of MLR that uses PCA when estimating regression coefficients $\hat{\mathbf{w}}$. It is a procedure used to overcome problems which arise when the exploratory variables are nearly co-linear. In PCR, instead of regressing the dependent variable $y$ on the independent variables $x$ directly, the principal components $V_x$ of the independent variables are used. One typically only uses a subset of the principal components in the regression, making a kind of regularized estimation. Often the principal components with the highest variance are selected. A larger class of multivariate analysis methods that introduces a latent model into the standard linear regression is called latent variable regression (LVR) [55].

In the same way as MLR, we assume that the projection matrix $W$ is with rank 1, namely $W = w_y w_y^T$. A rank-$K$ approximation $\hat{X}$ of the data matrix $X$ can be obtained by singular value decomposition as

$$
\hat{X} = U_K \Sigma_K V_K^T, \tag{9}
$$

where $\Sigma_K$ is a $K \times K$ diagonal matrix whose diagonal components are top-$K$ eigenvalues obtained by PCA of $X$, and $V_K$ is a $K \times d_L$ matrix whose columns are the top-$K$ eigenvectors. Then, the objective function of PCR to be minimized can be obtained by substituting $\hat{X}$ into $X$ in the objective function of MLR, as follows:

$$
\tilde{C}^{(PCR)}(w_x, w_y) = \tilde{C}^{(MLR)}(w_x, w_y)\hat{X}, Y
$$

$$
\hat{E} \left[ w_y^T (y - \alpha w_x w_x^T y) w_x \right]
$$

$$
= \hat{E} \left[ y^T y - 2 \alpha \Sigma_{xy} \Sigma_{yy} w_x w_x^T + \alpha^2 \Sigma_{xx} \Sigma_{yy} w_x w_x^T \right]
$$

$$
= \hat{E} \left[ y^T y - 2 \alpha \Sigma_{xy} \Sigma_{yy} w_x w_x^T + \alpha^2 \Sigma_{xx} w_x w_x^T \right]
$$

$$
S_{yy} = \frac{1}{N} \sum_{i=1}^{N} u_{K,i} \Sigma_K V_K^T y_i y_i^T,
$$

$$
S_{xx} = \frac{1}{N} \sum_{i=1}^{N} u_{K,i} \Sigma_K V_K^T V_K \Sigma_K u_{K,i},
$$

$$
= \frac{1}{N} \sum_{i=1}^{N} u_{K,i} (\Sigma_K)^2 u_{K,i}.
$$

From the description of the previous subsection, we can obtain

$$
\hat{C}^{(PCR)} = \begin{pmatrix} 0 & S_{xy} \\ S_{yx} & 0 \end{pmatrix},
\hat{C}^{(PCR)} = \begin{pmatrix} S_{xx} + \delta I_{d_x} & 0 \\ 0 & I_{d_y} \end{pmatrix},
$$

$$
w = (w_x^T, w_y^T)^T.
$$

### 4.5 Partial Least Squares (PLS)

Partial Least Squares (PLS) [20] (or sometimes called PLS regression) belongs to a family of latent variable regression (LVR), and tries to find a direction for the observable sample set $X$ that explains the maximum variance direction for the predicted sample set $Y$. The contribution of PLS against the standard MLR and PCR is to simultaneously estimate the latent model and regression from the latent space to the predicted space, which leads to robust regression against noisy observations.

Although PLS cannot be formulated as a generalized eigenproblem in general, orthogonal PLS (OPLS) [56], [57] as a variant of the original PLS has a form of generalized eigenproblem. This improves the interpretability (but not the predictivity) of the original PLS. OPLS can be formulated as follows:

$$
XY^T X^T w = \lambda XX^T w,
$$

$$
w = (w_x^T, w_y^T)^T
$$

meaning,

$$
\hat{C}^{(OPLS)} = XX^T \propto S_{xy} S_{yy}^T,
\hat{C}^{(OPLS)} = XX^T \propto S_{xx}.
$$

When every sample $y_i$ in $Y$ represents a class indicator vectors (cf. Section 4.2), OPLS is called OPLS-discriminant analysis (OPLS-DA) [57], which has been often used for the task of bio-marker identification [58].

### 4.6 $\ell_2$-norm Regularization

$\ell_2$-norm regularization is a popular regularization technique for various optimization problems including multivariate analysis. In the area of statistics or machine learning, this is sometimes called Tikhonov regularization [9], [37]. The most popular method that utilizes $\ell_2$-norm regularization is ridge regression [36], which combines MLR and $\ell_2$-norm regularization. The objective function to be minimized is the following squared error:

$$
E^{(Ridge)}(w_x, w_y) = \hat{E} \left[ ||y - \alpha w_x w_x^T y||^2 + \delta ||w_x||^2 \right]
$$

$$
= \hat{E} \left[ (y - \alpha w_x w_x^T y)^2 \right] + \delta ||w_x||^2
$$

$$
= \hat{E} \left[ y^T y - 2 \alpha \Sigma_{xy} \Sigma_{yy} w_x w_x^T + 2 \alpha^2 \Sigma_{xx} \Sigma_{yy} w_x w_x^T \right] + \delta ||w_x||^2
$$

$$
= \hat{E} \left[ y^T y - 2 \alpha \Sigma_{xy} \Sigma_{yy} w_x w_x^T + 2 \alpha^2 \Sigma_{xx} (S_{xx} + \delta I_{d_x}) w_x w_x^T \right],
$$

where $\delta = \delta/\alpha^2$. From the above equation and the objective function of MLR, the GPE of ridge regression can be derived as

$$
\hat{C}^{(Ridge)} = \begin{pmatrix} 0 & S_{xy} \\ S_{yx} & 0 \end{pmatrix},
$$

$$
\hat{C}^{(Ridge)} = \begin{pmatrix} S_{xx} + \delta I_{d_x} & 0 \\ 0 & I_{d_y} \end{pmatrix},
$$

$$
w = (w_x^T, w_y^T)^T.
$$

In a similar way to ridge regression, we can derive the GPE of
CCA with $\ell_2$-norm regularization [38], [59] as

$$\mathbf{C}(\text{CCA}_{-\ell_2}) = \begin{pmatrix} 0 & \mathbf{S}_{xy} \\ \mathbf{S}_{yx} & 0 \end{pmatrix},$$

$$\mathbf{C}(\text{CCA}_{-\ell_2}) = \begin{pmatrix} \mathbf{S}_{xx} + \lambda \mathbf{I}_n & 0 \\ 0 & \mathbf{S}_{yy} + \lambda \mathbf{I}_n \end{pmatrix}.$$

In addition, we can incorporate $\ell_1$-norm regularization into the GPE framework only if the objective generalized eigenproblem has the following form:

$$\mathbf{X} \mathbf{L}_Q \mathbf{X}^\top \mathbf{w} = \lambda \mathbf{X} \mathbf{S}_Q \mathbf{X}^\top \mathbf{w},$$

meaning

$$\mathbf{S}_{Q-S} \mathbf{w} = \lambda \mathbf{S}_Q \mathbf{w}.$$
4.10 Semi-supervised CCA

In a similar way to that of SELF, a semi-supervised extension of CCA can be derived, which is called SemiCCA [44].

Assume that there are two samples sets $X$ and $Y$, and each includes incomplete sample set $X^{(I)}$ and $Y^{(I)}$, namely there are at least one unpaired samples in both $X$ and $Y$. SemiCCA integrates the GPE of CCA calculated only from the complete sample sets and the GPE of PCA calculated from the complete and incomplete sample sets, as follows:

\[
C_{\text{SemiCCA}}^{(I)} = \beta S_{xx}^{(I)} + (1 - \beta)I_{xx},
\]

\[
C_{\text{SemiCCA}} = \beta S_{xx} + (1 - \beta)I_{xx},
\]

where $\beta$ is a hyper parameter satisfying $0 \leq \beta \leq 1$. When $\beta = 1$, SemiCCA is equivalent to CCA only with the complete samples ($X^{(I)}, Y^{(I)}$). Meanwhile, when $\beta = 0$, SemiCCA is equivalent to PCA with all samples in $X$. Generally speaking, SELF inherits the properties of both LFDA and PCA, and their influences can be controlled by the parameter $\beta$.

6. Kernelized Extensions

6.1 Kernelization of Standard Methods

Almost all the methods in the GPE framework can be kernelized in a similar manner to the existing ones. First, we describe kernel CCA [34], [35] and related regularization techniques.

The original CCA can be extended to, e.g., non-vectorial domains by defining kernels over $x$ and $y$,

\[
k_x(x_i, x_j) = \langle \phi_x(x_i), \phi_x(x_j) \rangle,
\]

\[
k_y(y_i, y_j) = \langle \phi_y(y_i), \phi_y(y_j) \rangle,
\]

and searching for solutions that lie in the span of $\phi_x(x)$ and $\phi_y(y)$.

\[
w_x = \sum_{n=1}^{N} \alpha_n \phi_x(x_n), \quad w_y = \sum_{n=1}^{N} \beta_n \phi_y(y_n).
\]

In this setting, we use the following empirical scatter matrix,

\[
\hat{S}_{xy} = \sum_{n=1}^{N} \phi_x(x_n)\phi_y(y_n)^T.
\]

Denoting the Gram matrices defined by the samples as $K_x$ and $K_y$, we can obtain the solution from the following optimization problem with respect to coefficient vectors, $\alpha$ and $\beta$.

\[
\min_{\alpha, \beta} ||w_x^T \alpha - w_y^T \beta||^2 + \beta_c K_y \beta
\]

In the same way as CCA, the optimization can be achieved by solving the following generalized eigenvalue problem:

\[
C^{(\text{CCA})}(\alpha, \beta) = \lambda C^{(\text{CCA})}(\alpha, \beta).
\]

\[
C^{(\text{CCA})} = \begin{pmatrix} K_x & K_x K_y \\ K_y K_x & \beta_c K_y \end{pmatrix},
\]

\[
C^{(\text{CCA})}^{-1} = \begin{pmatrix} K_x^2 & 0 \\ 0 & K_y^2 \end{pmatrix}
\]

Although the bases $(w_x, w_y)$ cannot be explicitly obtained, the projection to those bases can be calculated with the help of the kernel trick:

\[
w_i^T \phi_x(x) = \sum_{n=1}^{N} \alpha_n k_x(x_n, x_i).
\]
where each column of the Gram matrix \( K_i \) has \( k \) non-zero values. This minimization can be solved via a linear system of equations. Once \( K_i \) is calculated, LLE next finds a base that minimizes
\[
\rho^{(\text{LLE})}(w_i|X) = \alpha^\top (I_N - K_i)^2 \alpha \quad \text{s.t.} \quad \alpha^\top \alpha = 1.
\]
Therefore, the GPE of LLE can be obtained as
\[
\overline{\mathcal{C}}^{(\text{LLE})} = (I_N - K_i)^2, \quad \overline{\mathcal{C}}^{(\text{LLE})} = I_N.
\]

6.3 Clustering Methods
With the benefit of the kernelized extension of the GPE framework, several clustering methods such as spectral clustering (SC) \([13]\), \([63]\), \([64]\) and normalized cuts (NC) \([65]\) can also be included in the class of multivariate analysis we are concerned with.
\[
\overline{\mathcal{C}}^{(\text{SC})} = L_x, \quad \overline{\mathcal{C}}^{(\text{NC})} = D_x^{-1/2} L_x D_x^{-1/2}, \quad \overline{\mathcal{C}}^{(\text{NC})} = I_N.
\]

Kernel k-means \([66]\) is also known to belong to this family if we admit introducing a certain iterative procedure \([67]\). The details can be seen in the preceding work by De la Torre \([49]\), \([50]\).

6.4 How to Design New Kernelized Methods
Integrating two methods within the kernelized GPE framework is not straightforward, since a simple addition of Gram matrices is not GPE.

For example, let us consider the following generalized eigenproblem characterized by two matrices of GPE-style second-order statistics:
\[
\begin{align*}
\overline{\mathcal{C}}w &= \lambda \overline{\mathcal{C}}w, \\
\overline{\mathcal{C}} &= \bar{X} L_x \bar{X}^\top + \overline{\mathcal{T}_2}, \quad \mathcal{C} = X L_x X^\top + L_2,
\end{align*}
\]
where \( \mathcal{T}_1 \) and \( L_1 \) are both positive semi-definite matrices and \( \mathcal{T}_2 \) and \( L_2 \) are both non-negative symmetric matrices, from the definition of GPE. The above generalized eigenproblem can be replaced with the following one without essentially changing the solution \([43]\)
\[
\begin{align*}
\overline{\mathcal{C}}w &= \lambda \overline{\mathcal{C}}w, \\
\overline{\mathcal{C}}_2 &= \bar{X}(L_1 + (X^\top L_2 X)^\top) X^\top = \bar{X} \mathcal{L}_2 X^\top, \\
\mathcal{C}_2 &= X(L_1 + (X^\top L_2 X)^\top) X^\top = X \mathcal{L}_2 X^\top,
\end{align*}
\]
where \(^\top\) denotes the Moore-Penrose pseudo-inverse \([68]\). In this derivation, we used the fact that the matrices \( \overline{\mathcal{C}} \) and \( \mathcal{C}_2 \) (resp. \( \mathcal{C} \) and \( \mathcal{C}_2 \)) that characterize the generalized eigenproblems share the same range. Following the procedure shown in Sugiyama et al. \([43]\), we can obtain a kernelized variant of the multivariate analysis method formulated by Eq. (12).

From the above discussion, we can see that when dealing with kernelized multivariate analysis, we have to explicitly derive GPEs of existing methods, and replace the data matrix with its Gram matrix.

7. Concluding Remarks
This paper provided a new expression of covariance matrices
and Gram matrices, which we call generalized pairwise expression (GPE). This provided a unified insight into various multivariate analysis methods and their extensions. GPE made it easy to design desired multivariate analysis methods by simple combinations of GPEs of existing methods as templates. According to this methodology, we designed several new multivariate analysis methods.

The GPE framework covers a wide variety of multivariate analysis methods, and thus the way we have presented in this paper for designing new methods is still one of the examples. Developing more general guidelines would be promising future work.

References


[48] Sun, L., Ji, S. and Ye: Canonical correlation analysis for multi-


